Vector space

Examples

(1) $\mathbb{R}^n$ w.r.t. addition and multiplication, similarly $\mathbb{C}$ and $\mathbb{C}^n$.

(2) vector space of matrices $M_{m\times n}(\mathbb{R})$, i.e $m\times n$ matrices with real entries.

(3) vector space of polynomials $P_n(\mathbb{R})$, i.e.

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n \qquad a_i \in \mathbb{R}$$

if $a_n \neq 0$, then $\deg(f) = n$.

Furthermore $P(\mathbb{R})$ polynomials of all degrees.

(4) $C[a,b]$: set of all continuous functions on $[a,b]$ where $a<b$ real numbers.

It is clear that $(f+g)(x) = f(x)+g(x)$ and by Calculus is continuous on $[a,b]$.

$$c f(x) = c(f(x)) \in C[a,b]$$

& $f(x) = 0$ is continuous on $[a,b]$

(5) $D[a,b]$ = consists of all differentiable functions on $[a,b]$.

(6) The set of real (complex) sequences $(x_1, x_2, \ldots)$ satisfying $\sum_{i=1}^{\infty} |x_i|^2 < \infty$ is called the real (complex)

ℓ_2 - space is a vector space.

the addition and scalar multiplication

$$(x_1, x_2, \ldots) + (y_1, y_2, \ldots) = (x_1+y_1, x_2+y_2, \ldots)$$

$$\alpha(x_1, x_2, \ldots) = (\alpha x_1, \alpha x_2, \ldots)$$

if $\mathbf{X} = (x_i) \in \ell_2$ & $\mathbf{Y} = (y_i) \in \ell_2$, then $\mathbf{X}+\mathbf{Y} \in \ell_2$, since

$$\sum_{i=1}^{\infty} |x_i + y_i|^2 \leq 2\sum_{i=1}^{\infty} |x_i|^2 + 2\sum_{i=1}^{\infty} |y_i|^2$$

also $\alpha \mathbf{X} \in \ell_2$, because $\sum_{i} |\alpha x_i|^2 = |\alpha|^2 \sum_{i} |x_i|^2 < \infty$

~~Examples (Subspaces)~~

lemma: If u and v are vectors in a vector space V, then the following statements are true:

ⓐ $0v = 0$ and $c0 = 0$ c = scalar.

ⓑ If $u + v = 0$, then $u = -v$

ⓒ $(-1)v = -v$.

Proof: using prop(9) let $c = d = 0$, we get

$$0v = 0v + 0v \quad \text{add } -(0v)$$

$$\Rightarrow -(0v) + 0v = [-0v + 0v] + 0v$$

thus implies that $0 = 0v$ *.

(b) add $(-v)$ to both sides and use the associative law.

(c) Using (9) and (10)

$$v + (-v) = 1v + (-1)v$$
$$= (1 + (-1))v = 0v = 0$$

hence $(-1)v = -v$ by (b).

Subspaces: A subset S of a vector space V is called a subspace of V if the following are true

(i) S contain the zero vector.

(ii) if $v \in S$, then $cv \in S$, c = scalar.

(iii) If u and $v \in S$, then $u + v \in S$.

Examples: (1) V = vector space, then V itself is a subspace. The zero vector is a subspace. We call them improper subspace.

(2) $S \subseteq \mathbb{R}^2$ consists of all columns of the form $\begin{bmatrix} 2t \\ 3t \end{bmatrix}$ t is a real number.

(3) $S = \{x : Ax = 0\}$ the set of solutions of homogeneous linear system. (Nullspace of A).

(4) $S = \{y : y'' + 5y' + 6y = 0$ define on $[a,b]\}$

It is clear that S is a subset of $C[a,b]$.

Def (linear combination of vectors).

let $v_1, v_2, \dots, v_k$ be vectors in V. If $c_1, \dots, c_k$ are any scalars, the vector

$$c_1 v_1 + \dots + c_k v_k$$

is a linear combination of $v_1, \dots, v_k$.

Ex: $v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$ two vectors in $\mathbb{R}^2$. the linear combination of v_1 and v_2 is

$$c_1 v_1 + c_2 v_2 = \begin{bmatrix} 2c_1 - 3c_2 \\ c_1 + 4c_2 \end{bmatrix}$$

Def (Span)

Span of X is a set of all linear combinations of vectors in X. $(\langle X \rangle)$

Thm If $X \neq \phi$ is a subset of V. Then $\langle X \rangle$ is a subspace of V.

We refer to $\langle X \rangle$ as the subspace generated by X.

Ex: let $A = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$, $B = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} -1 \\ 5 \\ 6 \end{bmatrix} \in \mathbb{R}^3$.

Determine whether C belongs to the subspace generated (spanned) by A and B. (just solve $\alpha A + \beta B = C$.)

Def (linear independence of vectors)

Let $v_1, \ldots, v_k$ vectors in V and $c_1, \ldots, c_k$ are scalars if

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0$$

implies that $c_1 = c_2 = \cdots = c_k = 0$. Then $v_1, \ldots, v_k$ are linearly independent.

Ex: Consider the homog. sys $Ax = 0$, $A = [u \ v \ w]$. If $|A| \neq 0$, then u, v, w are linearly independent and $Ax = 0$ has only the trivial solution. On the other hand if $|A| = 0$, then u, v, w are linearly dependent and $Ax = 0$ has a non-trivial solution.

Ex: Show that the polynomials $x+1$, $x+2$, x^2-1 are linearly independent in the vector space $P_3(\mathbb{R})$.

Ex: Show that the vectors $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -4 \end{bmatrix}$ are linearly dependent in $\mathbb{R}^2$?

Application to DE: (Wronskian)

Suppose $f_1, \ldots, f_n$ are functions whose first $(n-1)$ derivatives exist at all points of the interval $[a,b]$. Assume that

$$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0 \quad \text{on } [a,b]$$

now differentiate this equation $(n-1)$ times gives.

$$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0$$

$$c_1 f_1' + c_2 f_2' + \cdots + c_n f_n' = 0$$

$$\vdots$$

$$c_1 f_1^{(n-1)} + c_2 f_2^{(n-1)} + \cdots + c_n f_n^{(n-1)} = 0$$

$$\Rightarrow \begin{bmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & & f_n^{(n-1)} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

If the determinant is not identically equal to zero on $[a,b]$, then $f_1, \ldots, f_n$ are linearly independent.

Define Wronskian: $W = \begin{vmatrix} f_1 & \cdots & f_n \\ \vdots & & \vdots \\ f_1^{(n-1)} & & f_n^{(n-1)} \end{vmatrix}$

Thm: Suppose that $f_1, \ldots, f_n$ are functions whose first $(n-1)$ derivatives exist in $[a,b]$. If $W(f_1, \ldots, f_n)$ is not identically zero on $[a,b]$, then $f_1, \ldots, f_n$ are linearly independent in $[a,b]$.

Remark: In general one can't conclude that if $f_1, \ldots, f_n$ are linearly independent, then $W(f_1, \ldots f_n)$ is not the zero function. However if $f_1, \ldots, f_n$ are solutions of a homog linear DE of order n, then $W(f_1, \ldots, f_n)$ can never vanish.

Ex: Show that x, e^{x}, e^{-2x} are linearly independent in the vector space $C[0,1]$.

Def (Basis)

Let $X \neq \phi$ subset of a vector space V. Then X is called a basis of V if the following hold:

(i) X is linearly independent

(ii) X span (generates) V.

Ex: $E_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, E_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$ are basis for $\mathbb{R}^n$.

i.e. for the Euclidean space $\mathbb{R}^n$. This is called standard basis for $\mathbb{R}^n$.