FUNCTIONS OF DIAGONALIZABLE MATRICES

For square matrices $\mathbf{A}$, what should it mean to write $\sin \mathbf{A}$, $e^{\mathbf{A}}$, $\ln \mathbf{A}$, etc.? A naive approach might be to simply apply the given function to each entry of $\mathbf{A}$ such as

$$\sin \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} \sin a_{11} & \sin a_{12} \\ \sin a_{21} & \sin a_{22} \end{pmatrix}. \tag{7.3.1}$$

But doing so results in matrix functions that fail to have the same properties as their scalar counterparts. For example, since $\sin^2 x + \cos^2 x = 1$ for all scalars x, we would like our definitions of $\sin \mathbf{A}$ and $\cos \mathbf{A}$ to result in the analogous matrix identity $\sin^2 \mathbf{A} + \cos^2 \mathbf{A} = \mathbf{I}$ for all square matrices $\mathbf{A}$. The entrywise approach (7.3.1) clearly fails in this regard.

One way to define matrix functions possessing properties consistent with their scalar counterparts is to use infinite series expansions. For example, consider the exponential function

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} \cdots. \tag{7.3.2}$$

Formally replacing the scalar argument z by a square matrix $\mathbf{A}$ ($z^0 = 1$ is replaced with $\mathbf{A}^0 = \mathbf{I}$) results in the infinite series of matrices

$$e^{\mathbf{A}} = \mathbf{I} + \mathbf{A} + \frac{\mathbf{A}^2}{2!} + \frac{\mathbf{A}^3}{3!} \cdots, \tag{7.3.3}$$

called the *matrix exponential*. While this results in a matrix that has properties analogous to its scalar counterpart, it suffers from the fact that convergence must be dealt with, and then there is the problem of describing the entries in the limit. These issues are handled by deriving a closed form expression for (7.3.3).

If $\mathbf{A}$ is diagonalizable, then $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} = \mathbf{P} \operatorname{diag}(\lambda_1, \ldots, \lambda_n) \mathbf{P}^{-1}$, and $\mathbf{A}^k = \mathbf{P}\mathbf{D}^k\mathbf{P}^{-1} = \mathbf{P} \operatorname{diag}\left(\lambda_1^k, \ldots, \lambda_n^k\right) \mathbf{P}^{-1}$, so

$$e^{\mathbf{A}} = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{k!} = \sum_{k=0}^{\infty} \frac{\mathbf{P}\mathbf{D}^k\mathbf{P}^{-1}}{k!} = \mathbf{P} \left(\sum_{k=0}^{\infty} \frac{\mathbf{D}^k}{k!} \right) \mathbf{P}^{-1} = \mathbf{P} \operatorname{diag}\left(e^{\lambda_1}, \ldots, e^{\lambda_n}\right) \mathbf{P}^{-1}.$$

In other words, we don't have to use the infinite series (7.3.3) to define $e^{\mathbf{A}}$. Instead, define $e^{\mathbf{D}} = \operatorname{diag}(e^{\lambda_1}, e^{\lambda_2}, \ldots, e^{\lambda_n})$, and set

$$e^{\mathbf{A}} = \mathbf{P}e^{\mathbf{D}}\mathbf{P}^{-1} = \mathbf{P} \operatorname{diag}(e^{\lambda_1}, e^{\lambda_2}, \ldots, e^{\lambda_n}) \mathbf{P}^{-1}.$$

This idea can be generalized to any function $f(z)$ that is defined on the eigenvalues λ_i of a diagonalizable matrix $\mathbf{A} = \mathbf{PDP}^{-1}$ by defining $f(\mathbf{D})$ to be $f(\mathbf{D}) = \operatorname{diag}(f(\lambda_1), f(\lambda_2), \ldots, f(\lambda_n))$ and by setting

$$f(\mathbf{A}) = \mathbf{P}f(\mathbf{D})\mathbf{P}^{-1} = \mathbf{P}\operatorname{diag}(f(\lambda_1), f(\lambda_2), \ldots, f(\lambda_n))\mathbf{P}^{-1}. \tag{7.3.4}$$

At first glance this definition seems to have an edge over the infinite series approach because there are no convergence issues to deal with. But convergence worries have been traded for uniqueness worries. Because P is not unique, it's not apparent that (7.3.4) is well defined. The eigenvector matrix P you compute for a given A need not be the same as the eigenvector matrix I compute, so what insures that your f (A) will be the same as mine? The spectral theorem (p. 517) does. Suppose there are k distinct eigenvalues that are grouped according to repetition, and expand (7.3.4) just as (7.2.11) is expanded to produce

$$f(\mathbf{A}) = \mathbf{PDP}^{-1} = \left(\mathbf{X}_1|\mathbf{X}_2|\cdots|\mathbf{X}_k\right)\begin{pmatrix} f(\lambda_1)\mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & f(\lambda_2)\mathbf{I} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & f(\lambda_k)\mathbf{I} \end{pmatrix}\begin{pmatrix} \mathbf{Y}_1^T \\ \hline \mathbf{Y}_2^T \\ \hline \vdots \\ \hline \mathbf{Y}_k^T \end{pmatrix}$$

$$= \sum_{i=1}^{k} f(\lambda_i)\mathbf{X}_i\mathbf{Y}_i^T = \sum_{i=1}^{k} f(\lambda_i)\mathbf{G}_i.$$

Since $\mathbf{G}_i$ is the projector onto $N(\mathbf{A} - \lambda_i\mathbf{I})$ along $R(\mathbf{A} - \lambda_i\mathbf{I})$, $\mathbf{G}_i$ is uniquely determined by $\mathbf{A}$. Therefore, (7.3.4) uniquely defines $f(\mathbf{A})$ regardless of the choice of $\mathbf{P}$. We can now make a formal definition.

Functions of Diagonalizable Matrices

Let $\mathbf{A} = \mathbf{PDP}^{-1}$ be a diagonalizable matrix where the eigenvalues in $\mathbf{D} = \operatorname{diag}(\lambda_1\mathbf{I}, \lambda_2\mathbf{I}, \ldots, \lambda_k\mathbf{I})$ are grouped by repetition. For a function $f(z)$ that is defined at each $\lambda_i \in \sigma(\mathbf{A})$, define

$$f(\mathbf{A}) = \mathbf{P}f(\mathbf{D})\mathbf{P}^{-1} = \mathbf{P}\begin{pmatrix} f(\lambda_1)\mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & f(\lambda_2)\mathbf{I} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & f(\lambda_k)\mathbf{I} \end{pmatrix}\mathbf{P}^{-1} \quad (7.3.5)$$

$$= f(\lambda_1)\mathbf{G}_1 + f(\lambda_2)\mathbf{G}_2 + \cdots + f(\lambda_k)\mathbf{G}_k, \quad (7.3.6)$$

where $\mathbf{G}_i$ is the i^{th} spectral projector as described on pp. 517, 529. The generalization to nondiagonalizable matrices is on p. 603.

The discussion of matrix functions was initiated by considering infinite series, so, to complete the circle, a formal statement connecting infinite series with (7.3.5) and (7.3.6) is needed. By replacing $\mathbf{A}$ by $\mathbf{PDP}^{-1}$ in $\sum_{n=0}^{\infty} c_n(\mathbf{A}-z_0\mathbf{I})^n$ and expanding the result, the following result is established.

Infinite Series

If $f(z) = \sum_{n=0}^{\infty} c_n(z-z_0)^n$ converges when $|z-z_0| < r$, and if $|\lambda_i - z_0| < r$ for each eigenvalue λ_i of a diagonalizable matrix $\mathbf{A}$, then

$$f(\mathbf{A}) = \sum_{n=0}^{\infty} c_n(\mathbf{A}-z_0\mathbf{I})^n. \quad (7.3.7)$$

It can be argued that the matrix series on the right-hand side of (7.3.7) converges if and only if $|\lambda_i - z_0| < r$ for each λ_i, regardless of whether or not $\mathbf{A}$ is diagonalizable. So (7.3.7) serves to define $f(\mathbf{A})$ for functions with series expansions regardless of whether or not $\mathbf{A}$ is diagonalizable. More is said in Example 7.9.3 (p. 605).

Neumann Series Revisited. The function $f(z) = (1-z)^{-1}$ has the geometric series expansion $(1-z)^{-1} = \sum_{k=1}^{\infty} z^k$ that converges if and only if $|z| < 1$. This means that the associated matrix function $f(\mathbf{A}) = (\mathbf{I}-\mathbf{A})^{-1}$ is given by

$$(\mathbf{I}-\mathbf{A})^{-1} = \sum_{k=0}^{\infty} \mathbf{A}^k \quad \text{if and only if } |\lambda| < 1 \text{ for all } \lambda \in \sigma(\mathbf{A}). \quad (7.3.8)$$

This is the ***Neumann series*** discussed on p. 126, where it was argued that if $\lim_{n\to\infty} \mathbf{A}^n = \mathbf{0}$, then $(\mathbf{I}-\mathbf{A})^{-1} = \sum_{k=0}^{\infty} \mathbf{A}^k$. The two approaches are the same because it turns out that $\lim_{n\to\infty} \mathbf{A}^n = \mathbf{0} \iff |\lambda| < 1$ for all $\lambda \in \sigma(\mathbf{A})$. This is immediate for diagonalizable matrices, but the nondiagonalizable case is a bit more involved—the complete statement is developed on p. 618. Because $\max_i |\lambda_i| \leq \|\mathbf{A}\|$ for all matrix norms (Example 7.1.4, p. 497), a corollary of (7.3.8) is that $(\mathbf{I}-\mathbf{A})^{-1}$ exists and

$$(\mathbf{I}-\mathbf{A})^{-1} = \sum_{k=0}^{\infty} \mathbf{A}^k \quad \text{when } \|\mathbf{A}\| < 1 \text{ for any matrix norm.} \tag{7.3.9}$$

Caution! $(\mathbf{I}-\mathbf{A})^{-1}$ can exist without the Neumann series expansion being valid because all that's needed for $\mathbf{I}-\mathbf{A}$ to be nonsingular is $1 \notin \sigma(\mathbf{A})$, while convergence of the Neumann series requires each $|\lambda| < 1$.

Eigenvalue Perturbations. It's often important to understand how the eigenvalues of a matrix are affected by perturbations. In general, this is a complicated issue, but for diagonalizable matrices the problem is more tractable.

Problem: Suppose $\mathbf{B} = \mathbf{A}+\mathbf{E}$, where $\mathbf{A}$ is diagonalizable, and let $\beta \in \sigma(\mathbf{B})$. If $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D} = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, explain why

$$\min_{\lambda_i \in \sigma(\mathbf{A})} |\beta - \lambda_i| \leq \kappa(\mathbf{P})\,\|\mathbf{E}\|, \quad \text{where} \quad \kappa(\mathbf{P}) = \|\mathbf{P}\|\,\|\mathbf{P}^{-1}\| \tag{7.3.10}$$

for matrix norms satisfying $\|\mathbf{D}\| = \max_i |\lambda_i|$ (e.g., any standard induced norm).

Solution: Assume $\beta \notin \sigma(\mathbf{A})$—(7.3.10) is trivial if $\beta \in \sigma(\mathbf{A})$—and observe that

$$(\beta\mathbf{I}-\mathbf{A})^{-1}(\beta\mathbf{I}-\mathbf{B}) = (\beta\mathbf{I}-\mathbf{A})^{-1}(\beta\mathbf{I}-\mathbf{A}-\mathbf{E}) = \mathbf{I}-(\beta\mathbf{I}-\mathbf{A})^{-1}\mathbf{E}$$

implies that $1 \leq \|(\beta\mathbf{I}-\mathbf{A})^{-1}\mathbf{E}\|$—otherwise $\mathbf{I}-(\beta\mathbf{I}-\mathbf{A})^{-1}\mathbf{E}$ is nonsingular by (7.3.9), which is impossible because $(\beta\mathbf{I}-\mathbf{B})$ (and hence $(\beta\mathbf{I}-\mathbf{A})^{-1}(\beta\mathbf{I}-\mathbf{B})$ is singular). Consequently,

$$\begin{aligned} 1 &\leq \|(\beta\mathbf{I}-\mathbf{A})^{-1}\mathbf{E}\| = \|\mathbf{P}(\beta\mathbf{I}-\mathbf{D})^{-1}\mathbf{P}^{-1}\mathbf{E}\| \leq \|\mathbf{P}\|\,\|(\beta\mathbf{I}-\mathbf{D})^{-1}\|\,\|\mathbf{P}^{-1}\|\,\|\mathbf{E}\| \\ &= \kappa(\mathbf{P})\,\|\mathbf{E}\| \max_i |\beta-\lambda_i|^{-1} = \kappa(\mathbf{P})\,\|\mathbf{E}\| \frac{1}{\min_i |\beta-\lambda_i|}, \end{aligned}$$

and this produces (7.3.10). Similar to the case of linear systems (Example 5.12.1, p. 414), the expression $\kappa(\mathbf{P})$ is a *condition number* in the sense that if $\kappa(\mathbf{P})$ is relatively small, then the λ_i's are relatively insensitive, but if $\kappa(\mathbf{P})$ is relatively large, we must be suspicious. **Note:** Because it's a corollary of their 1960 results, the bound (7.3.10) is often referred to as the ***Bauer–Fike bound***.

Infinite series representations can always be avoided because *every function of* $\mathbf{A}_{n\times n}$ *can be expressed as a polynomial in* $\mathbf{A}$. In other words, when $f(\mathbf{A})$ exists, there is a polynomial $p(z)$ such that $p(\mathbf{A}) = f(\mathbf{A})$. This is true for all matrices, but the development here is limited to diagonalizable matrices—nondiagonalizable matrices are treated in Exercise 7.3.7. In the diagonalizable case, $f(\mathbf{A})$ exists if and only if $f(\lambda_i)$ exists for each $\lambda_i \in \sigma(\mathbf{A}) = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$, and, by (7.3.6), $f(\mathbf{A}) = \sum_{i=1}^{k} f(\lambda_i)\mathbf{G}_i$, where $\mathbf{G}_i$ is the i^{th} spectral projector. Any polynomial $p(z)$ agreeing with $f(z)$ on $\sigma(\mathbf{A})$ does the job because if $p(\lambda_i) = f(\lambda_i)$ for each $\lambda_i \in \sigma(\mathbf{A})$, then

$$p(\mathbf{A}) = \sum_{i=1}^{k} p(\lambda_i)\mathbf{G}_i = \sum_{i=1}^{k} f(\lambda_i)\mathbf{G}_i = f(\mathbf{A}).$$

But is there always a polynomial satisfying $p(\lambda_i) = f(\lambda_i)$ for each $\lambda_i \in \sigma(\mathbf{A})$? Sure—that's what the *Lagrange interpolating polynomial* from Example 4.3.5 (p. 186) does. It's given by

$$p(z) = \sum_{i=1}^{k} \left(f(\lambda_i) \frac{\prod_{\substack{j=1 \\ j\neq i}}^{k} (z - \lambda_j)}{\prod_{\substack{j=1 \\ j\neq i}}^{k} (\lambda_i - \lambda_j)} \right), \text{ so } f(\mathbf{A}) = p(\mathbf{A}) = \sum_{i=1}^{k} \left(f(\lambda_i) \frac{\prod_{\substack{j=1 \\ j\neq i}}^{k} (\mathbf{A} - \lambda_j\mathbf{I})}{\prod_{\substack{j=1 \\ j\neq i}}^{k} (\lambda_i - \lambda_j)} \right).$$

Using the function $g_i(z) = \begin{cases} 1 & \text{if } z = \lambda_i, \\ 0 & \text{if } z \neq \lambda_i, \end{cases}$ with this representation as well as that in (7.3.6) yields $\prod_{\substack{j=1 \\ j\neq i}}^{k} (\mathbf{A} - \lambda_j\mathbf{I}) \Big/ \prod_{\substack{j=1 \\ j\neq i}}^{k} (\lambda_i - \lambda_j) = g_i(\mathbf{A}) = \mathbf{G}_i$. For example, if $\sigma(\mathbf{A}_{n\times n}) = \{\lambda_1, \lambda_2, \lambda_3\}$, then $f(\mathbf{A}) = f(\lambda_1)\mathbf{G}_1 + f(\lambda_2)\mathbf{G}_2 + f(\lambda_3)\mathbf{G}_3$ with

$$\mathbf{G}_1 = \frac{(\mathbf{A}-\lambda_2\mathbf{I})(\mathbf{A}-\lambda_3\mathbf{I})}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}, \quad \mathbf{G}_2 = \frac{(\mathbf{A}-\lambda_1\mathbf{I})(\mathbf{A}-\lambda_3\mathbf{I})}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}, \quad \mathbf{G}_3 = \frac{(\mathbf{A}-\lambda_1\mathbf{I})(\mathbf{A}-\lambda_2\mathbf{I})}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}.$$

Below is a summary of these observations.

Spectral Projectors

If $\mathbf{A}$ is diagonalizable with $\sigma(\mathbf{A}) = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$, then the spectral projector onto $N(\mathbf{A} - \lambda_i \mathbf{I})$ along $R(\mathbf{A} - \lambda_i \mathbf{I})$ is given by

$$\mathbf{G}_i = \prod_{\substack{j=1 \\ j \neq i}}^{k} (\mathbf{A} - \lambda_j \mathbf{I}) \Big/ \prod_{\substack{j=1 \\ j \neq i}}^{k} (\lambda_i - \lambda_j) \quad \text{for } i = 1, 2, \ldots, k. \tag{7.3.11}$$

Consequently, if $f(z)$ is defined on $\sigma(\mathbf{A})$, then $f(\mathbf{A}) = \sum_{i=1}^{k} f(\lambda_i)\mathbf{G}_i$ is a polynomial in $\mathbf{A}$ of degree at most $k-1$.

Problem: For a scalar t, determine the matrix exponential $e^{\mathbf{A}t}$, where

$$\mathbf{A} = \begin{pmatrix} -\alpha & \beta \\ \alpha & -\beta \end{pmatrix} \quad \text{with } \alpha + \beta \neq 0.$$

Solution 1: The characteristic equation for $\mathbf{A}$ is $\lambda^2 + (\alpha + \beta)\lambda = 0$, so the eigenvalues of $\mathbf{A}$ are $\lambda_1 = 0$ and $\lambda_2 = -(\alpha + \beta)$. Note that $\mathbf{A}$ is diagonalizable

because no eigenvalue is repeated—recall (7.2.6). Using the function $f(z) = e^{zt}$, the spectral representation (7.3.6) says that

$$e^{\mathbf{A}t} = f(\mathbf{A}) = f(\lambda_1)\mathbf{G}_1 + f(\lambda_2)\mathbf{G}_2 = e^{\lambda_1 t}\mathbf{G}_1 + e^{\lambda_2 t}\mathbf{G}_2.$$

The spectral projectors $\mathbf{G}_1$ and $\mathbf{G}_2$ are determined from (7.3.11) to be

$$\mathbf{G}_1 = \frac{\mathbf{A} - \lambda_2 \mathbf{I}}{-\lambda_2} = \frac{1}{\alpha + \beta}\begin{pmatrix} \beta & \beta \\ \alpha & \alpha \end{pmatrix} \quad \text{and} \quad \mathbf{G}_2 = \frac{\mathbf{A}}{\lambda_2} = \frac{1}{\alpha + \beta}\begin{pmatrix} \alpha & -\beta \\ -\alpha & \beta \end{pmatrix},$$

so

$$e^{\mathbf{A}t} = \mathbf{G}_1 + e^{-(\alpha+\beta)t}\mathbf{G}_2 = \frac{1}{\alpha + \beta}\left[\begin{pmatrix} \beta & \beta \\ \alpha & \alpha \end{pmatrix} + e^{-(\alpha+\beta)t}\begin{pmatrix} \alpha & -\beta \\ -\alpha & \beta \end{pmatrix}\right].$$

Solution 2: Compute eigenpairs $(\lambda_1, \mathbf{x}_1)$ and $(\lambda_2, \mathbf{x}_2)$, construct $\mathbf{P} = \begin{bmatrix} \mathbf{x}_1 \,|\, \mathbf{x}_2 \end{bmatrix}$, and compute

$$e^{\mathbf{A}t} = \mathbf{P} \begin{pmatrix} f(\lambda_1) & 0 \\ 0 & f(\lambda_2) \end{pmatrix} \mathbf{P}^{-1} = \mathbf{P} \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \mathbf{P}^{-1}.$$

The computational details are called for in Exercise 7.3.2.

Problem: For $\mathbf{T} = \begin{pmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{pmatrix}$, evaluate $\lim_{k \to \infty} \mathbf{T}^k$.

Solution 1: Compute two eigenpairs, $\lambda_1 = 1$, $\mathbf{x}_1 = (1,\ 1)^T$, and $\lambda_2 = 1/4$, $\mathbf{x}_2 = (-2,\ 1)^T$. If $\mathbf{P} = [\mathbf{x}_1 \,|\, \mathbf{x}_2]$, then $\mathbf{T} = \mathbf{P} \begin{pmatrix} 1 & 0 \\ 0 & 1/4 \end{pmatrix} \mathbf{P}^{-1}$, so

$$\mathbf{T}^k = \mathbf{P} \begin{pmatrix} 1^k & 0 \\ 0 & 1/4^k \end{pmatrix} \mathbf{P}^{-1} \to \mathbf{P} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{P}^{-1} = \frac{1}{3} \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}. \tag{7.3.12}$$

Solution 2: We know from (7.3.6) that $\mathbf{T}^k = 1^k \mathbf{G}_1 + (1/4)^k \mathbf{G}_2 \to \mathbf{G}_1$. Since $\lambda_1 = 1$ is a simple eigenvalue, formula (7.2.12) on p. 518 can be used to compute $\mathbf{G}_1 = \mathbf{x}_1 \mathbf{y}_1^T / \mathbf{y}_1^T \mathbf{x}_1$, where $\mathbf{x}_1$ and $\mathbf{y}_1^T$ are any right- and left-hand eigenvectors associated with $\lambda_1 = 1$. A right-hand eigenvector $\mathbf{x}_1$ was computed above. Computing a left-hand eigenvector $\mathbf{y}_1^T = (1,\ 2)$ yields

$$\mathbf{T}^k \to \mathbf{G}_1 = \frac{\mathbf{x}_1 \mathbf{y}_1^T}{\mathbf{y}_1^T \mathbf{x}_1} = \frac{1}{3} \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}. \tag{7.3.13}$$