URV Factorization

$A \in \mathbb{R}^{m\times n}$ of rank=r, there are orthogonal matrices $U_{m\times m}$ and $V_{n\times n}$ and a nonsingular matrix $C_{r\times r}$ s.th.

$$A = URV^T = U \begin{pmatrix} C_{r\times r} & 0 \\ 0 & 0 \end{pmatrix}_{m\times n} V^T$$

1) The first r columns in U are an orthonormal basis for $R(A)$.
2) The last m-r Columns of U are an orthonormal basis for $N(A^T)$.
3) The first r columns in V are an orthonormal basis for $R(A^T)$
4) The last (n-r) columns of V are an orthonormal basis for $N(A)$.

<u>Thm:</u> For rank$(A_{n\times n})=r$, the following are equivalent

(1) $R(A) \perp N(A)$

(2) $R(A) = R(A^T)$

(3) $N(A) = N(A^T)$

(4) $A = U \begin{pmatrix} C_{r\times r} & 0 \\ 0 & 0 \end{pmatrix} U^T$, U orthogonal & C nonsingular.

<u>Proof:</u> $(1) \leftrightarrow (2) \longleftrightarrow (3)$ is direct from last lecture.

let us prove that $(4) \overset{?}{\longleftrightarrow} (2)$

if (4) is the URV factorization with $V = U = (U_1\ U_2)$ then $R(A) = R(U_1) = R(V_1) = R(A^T)$. Conversely if $R(A) = R(A^T)$ perping both side and using the equation $\{R(A)^\perp = N(A^T)$ & $N(A)^\perp = R(A^T)\}$ produces $N(A) = N(A^T)$, so $U = (u_1 \cdots u_m)$ & $V = (v_1 \cdots v_n)$ yields URV Factorization with $U = V$.