Matrices

1.2.1 Notation for a Matrix and Operations with Matrices

The vector space of all $m \times n-$matrices with real elements will be denoted by $\mathbf{R}^{m\times n}$ and

$$A \in \mathbf{R}^{m\times n} \Leftrightarrow A = (a_{ik}) = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}, \quad a_{ik} \in \mathbf{R}.$$

The element of the matrix A that stands in the $i-$th row and $k-$th column will be denoted by a_{ik} or $A(i,k)$ or $[A]_{ik}$. The main operations with matrices are following:

- transposition of matrices $(\mathbf{R}^{m\times n} \to \mathbf{R}^{n\times m})$

$$B = A^T \quad \Leftrightarrow \quad b_{ik} = a,$$

- addition of matrices $(\mathbf{R}^{m\times n} \times \mathbf{R}^{m\times n} \to \mathbf{R}^{m\times n})$

$$C = A + B \quad \Leftrightarrow \quad c_{ik} = a_{ik} + b_{ik},$$

- multiplication of matrices by a number $(\mathbf{R} \times \mathbf{R}^{m\times n} \to \mathbf{R}^{m\times n})$

$$B = \lambda A \quad \Leftrightarrow \quad b_{ik} = \lambda a_{ik}$$

- multiplication of matrices $(\mathbf{R}^{m\times p} \times \mathbf{R}^{p\times n} \to \mathbf{R}^{m\times n})$

$$C = AB \quad \Leftrightarrow \quad c_{ik} = \sum_{j=1}^{p} a_{ij} b_{jk}.$$

Problem 2.1.1.* Let

$$A = \begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix}, \quad B = \begin{bmatrix} k & n \\ l & p \\ m & r \end{bmatrix}.$$

Find the matrix AB.

Problem 2.1.3.* Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Prove that

$$A^n = 2^{n-1}A \ \ (n \in N).$$

Example 2.1.1.* Let us show that multiplication of matrices is not commutative. Let

$$A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}, \ \ B = \begin{bmatrix} -2 & 5 \\ 1 & 2 \end{bmatrix}.$$

We find the products:

$$AB = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -2 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 13 \\ -4 & 19 \end{bmatrix},$$

$$BA = \begin{bmatrix} -2 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 13 & 2 \\ 7 & 8 \end{bmatrix}.$$

As $AB \neq BA$ does not hold for the example, multiplication of matrices is not commutative in general.

Proposition 2.1.1. If $A \in \mathbf{R}^{m\times p}$ and $B \in \mathbf{R}^{p\times n}$, then

$$(AB)^T = B^T A^T.$$

Proof. If $C = (AB)^T$, then

$$c_{ik} = [(AB)^T]_{ik} = [AB]_{ki} = \sum_{j=1}^{p} a_{kj} b_{ji}.$$

If $D = B^T A^T$, we also have

$$d_{ik} = [B^T A^T]_{ik} = \sum_{j=1}^{p} [B^T]_{ij} [A^T]_{jk} = \sum_{j=1}^{p} [B]_{ji} [A]_{kj} =$$

$$= \sum_{j=1}^{p} a_{kj} b_{ji} = c_{ik}. \ \ \square$$

Definition 2.1.1. A matrix $A \in \mathbf{R}^{n\times n}$ is called *symmetric* if $A^T = A$ and *skew-symmetric* if $A^T = -A$.

Problem 2.1.4.* Is matrix A symmetric or skew-symmetric if

$$a)\ A = \begin{bmatrix} -1 & 3 & 2 \\ 3 & 1 & 3 \\ 2 & 3 & -1 \end{bmatrix},\ b)\ A = \begin{bmatrix} 0 & 2 & -4 \\ -2 & 1 & -7 \\ 4 & 7 & 2 \end{bmatrix},\ c)\ A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 1 & 2 \\ -5 & 1 & 4 \end{bmatrix}.$$

Proposition 2.1.2. Each matrix $A \in \mathbf{R}^{n\times n}$ can be expressed as a sum of a symmetric matrix and a skew-symmetric matrix.

Proof. Each matrix $A \in \mathbf{R}^{n\times n}$ can be expressed as $A = B + C$, where $B = (A + A^T)/2$ and $C = (A - A^T)/2$. As

$$B^T = ((A + A^T)/2)^T = (A^T + A)/2 = B$$

and

$$C^T = ((A - A^T)/2)^T = C = (A^T - A)/2 = -C,$$

the proposition holds. □

Problem 2.1.5.* Represent the matrix

$$A = \begin{bmatrix} 2 & -3 & 5 & 1 \\ -3 & -2 & 3 & 0 \\ 3 & -7 & 0 & 6 \\ 4 & 5 & 2 & 4 \end{bmatrix}$$

as a sum of a symmetric and a skew-symmetric matrix.

Definition 2.1.2. If A is a $m \times n-$matrix with complex elements, i.e., $A \in \mathbf{C}^{m\times n}$, then the *transposed skew-symmetric matrix* A^H will be defined by the equality

$$B = A^H \quad \Leftrightarrow \quad b_{ik} = \overline{a}_{ki}.$$

Definition 2.1.3. A matrix $A \in \mathbf{C}^{n\times n}$ is called an *Hermitian matrix* if

$$A^H = A.$$

Problem 2.1.6.* Is matrix A an Hermitian matrix if

$$a)\ A = \begin{bmatrix} i & -2+i & -5+3i \\ 2+i & 5i & -2+i \\ 5+3i & 2+i & -8i \end{bmatrix},\ b)\ A = \begin{bmatrix} 5 & 2+3i & 1+i \\ 2-3i & -3 & -2i \\ 1-i & 2i & 0 \end{bmatrix}.$$

Problem 2.1.7.* Let $A \in \mathbf{C}^{m\times n}$. Show that matrices AA^H and A^HA are Hermitian matrices.

The matrix $A \in \mathbf{C}^{m\times n}$ can be expressed both by the column-vectors $\mathbf{c}_k$ (k = 1 : n) of the matrix A and by the row-vectors $\mathbf{r}_i^T$ (i = 1 : m) of the transpose of matrix A ("pasting" the matrices of the column-vectors or of the transposed row-vectors)

$$A = \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}_n \end{bmatrix} \equiv \begin{bmatrix} \mathbf{c}_1, & \cdots & , \mathbf{c}_n \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1^T \\ \vdots \\ \mathbf{r}_m^T \end{bmatrix},$$

where $\mathbf{c}_k \in \mathbf{C}^m$ and $\mathbf{r}_i \in \mathbf{C}^n$ and

$$\mathbf{r}_i = \begin{bmatrix} a_{i1} \\ \vdots \\ a_{in} \end{bmatrix}, \quad \mathbf{c}_k = \begin{bmatrix} a_{1k} \\ \vdots \\ a_{mk} \end{bmatrix}.$$

Example 2.1.2. Let us demonstrate these notions on a matrix $A \in \mathbf{R}^{3\times 2}$:

$$\begin{aligned}
A &= \begin{bmatrix} 2 & 3 \\ 4 & 1 \\ 3 & 2 \end{bmatrix} \Rightarrow \mathbf{c}_1 = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix} \wedge \mathbf{c}_2 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \wedge \\
\mathbf{r}_1 &= \begin{bmatrix} 2 \\ 3 \end{bmatrix} \wedge \mathbf{r}_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \wedge \mathbf{r}_3 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \wedge \\
\mathbf{r}_1^T &= \begin{bmatrix} 2 & 3 \end{bmatrix} \wedge \mathbf{r}_2^T = \begin{bmatrix} 4 & 1 \end{bmatrix} \wedge \mathbf{r}_3 = \begin{bmatrix} 3 & 2 \end{bmatrix} \wedge \\
A &= \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{c}_1, & \mathbf{c}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1^T \\ \mathbf{r}_2^T \\ \mathbf{r}_3^T \end{bmatrix}.
\end{aligned}$$

If $A \in \mathbf{R}^{m\times n}$, then $A(i,:)$ denotes the $i-$th row of the matrix A, i.e.,

$$A(i,:) = \begin{bmatrix} a_{i1} & \cdots & a_{in} \end{bmatrix},$$

and $A(:,\, k)$ denotes the k-th column of the matrix A, i.e.,

$$A(:,\, k) = \begin{bmatrix} a_{1k} \\ \vdots \\ a_{mk} \end{bmatrix}.$$

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If $1 \le p \le q < n \ \wedge \ 1 \le r \le m$, then

$$A(r,\, p : q) = \begin{bmatrix} a_{rp} & \cdots & a_{rq} \end{bmatrix} \in \mathbf{R}^{1\times(q-p+1)}$$

and if $1 \le p \le n \ \wedge \ 1 \le r \le s \le m$, then

$$A(r : s,\, p) = \begin{bmatrix} a_{rp} \\ \vdots \\ a_{sp} \end{bmatrix} \in \mathbf{R}^{s-r+1}.$$

If $A \in \mathbf{R}^{m\times n}$ and $\mathbf{i} = (i_1, \dots, i_p)$ and $\mathbf{k} = (k_1, \dots, k_q)$, where

$$i_1, \dots, i_p \in \{1; 2; \dots; m\} \quad \wedge \quad k_1, \dots, k_q \in \{1; 2; \dots; n\},$$

then the corresponding *submatrix* is

$$A(\mathbf{i}, \mathbf{k}) = \begin{bmatrix} A(i_1, k_1) & \cdots & A(i_1, k_q) \\ \vdots & & \vdots \\ A(i_p, k_1) & \cdots & A(i_p, k_q) \end{bmatrix}.$$

Example 2.1.3. If

$$A = \begin{bmatrix} 1 & 4 & -1 & 2 & -4 & 8 \\ 2 & -2 & 4 & 1 & 3 & 5 \\ 5 & 6 & -7 & 2 & -1 & 9 \\ 4 & 5 & 6 & -4 & 9 & 1 \end{bmatrix}$$

and $\mathbf{i} = (2; 4)$ and $\mathbf{k} = (1; 3; 5)$, then

$$A(\mathbf{i}, \mathbf{k}) = \begin{bmatrix} 2 & 4 & 3 \\ 4 & 6 & 9 \end{bmatrix}.$$

Example 2.2.5.* Let us find the product AB of block matrices A and B, when A and B are $3 \times 3-$matrices

$$A = \begin{bmatrix} 1 & 2 & \vdots & 2 \\ 3 & 4 & \vdots & 0 \\ \cdots & \cdots & \vdots & \cdots \\ 0 & 0 & \vdots & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & 1 & 0 & \vdots & 1 \\ 2 & 3 & -1 & \vdots & 1 \\ \cdots & \cdots & \cdots & \vdots & \cdots \\ 0 & 0 & 0 & \vdots & 1 \end{bmatrix}.$$

We denote

$$A = \begin{bmatrix} C & D \\ E & F \end{bmatrix}, \; B = \begin{bmatrix} G & H \\ K & L \end{bmatrix},$$

where

$$C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \; D = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \; E = \begin{bmatrix} 0 & 0 \end{bmatrix}, \; F = \begin{bmatrix} -1 \end{bmatrix}$$

and

$$G = \begin{bmatrix} -3 & 1 & 0 \\ 2 & 3 & -1 \end{bmatrix}, \; H = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; K = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}, \; L = \begin{bmatrix} 1 \end{bmatrix}.$$

We note that the dimensions of the matrices are in accordance with the conditions of multiplication of block matrices. If we denote

$$AB = \begin{bmatrix} R & S \\ T & U \end{bmatrix},$$

then

$$R = CG+DK = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -3 & 1 & 0 \\ 2 & 3 & -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 7 & -2 \\ -1 & 15 & -4 \end{bmatrix}$$

$$S = CH + DL = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix},$$

$$T = EG+FK = \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} -3 & 1 & 0 \\ 2 & 3 & -1 \end{bmatrix} + \begin{bmatrix} -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

and

$$U = EH + FL = \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} = \begin{bmatrix} -1 \end{bmatrix}.$$

Thus

$$AB = \begin{bmatrix} 1 & 7 & -2 & 5 \\ -1 & 15 & -4 & 7 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 7 & -2 & 5 \\ -1 & 15 & -4 & 7 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Problem 2.2.2.* Find the product AB of 4×5-matrix A and $5 \times 4-$matrix B in block form, when

$$A = \begin{bmatrix} 1 & 2 & 3 & \vdots & 0 & 0 \\ 0 & -1 & 4 & \vdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \vdots & 4 & 1 \\ 0 & 0 & 0 & \vdots & 7 & 5 \end{bmatrix}, \; B = \begin{bmatrix} 1 & -4 & \vdots & 0 & 0 \\ 2 & 3 & \vdots & 0 & 0 \\ 5 & -1 & \vdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \vdots & 1 & -1 \\ 0 & 0 & \vdots & 4 & -3 \end{bmatrix}.$$