The LU-Factorization

__Example:__ Consider $A = \begin{pmatrix} 2 & 2 & 2 \\ 4 & 7 & 7 \\ 6 & 18 & 22 \end{pmatrix}$

$$\begin{pmatrix} 2 & 2 & 2 \\ 4 & 7 & 7 \\ 6 & 18 & 22 \end{pmatrix} \xrightarrow[-3R_1+R_3]{-2R_1+R_2} \begin{pmatrix} 2 & 2 & 2 \\ 0 & 3 & 3 \\ 0 & 12 & 16 \end{pmatrix} \xrightarrow{-4R_2+R_3}$$

$$\to \begin{pmatrix} 2 & 2 & 2 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{pmatrix} = U$$

Define $G_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $G_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$ & $G_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{pmatrix}$

$$\Rightarrow \quad G_3 G_2 G_1 A = U$$

$$\Rightarrow \quad A = G_1^{-1} G_2^{-1} G_3^{-1} U = L U.$$

this is called LU factorization of A.

L = Lower matrix & U = upper matrix.

__Thm:__ The LU factorization is unique.

__Proof__ assume not $\Rightarrow$ $A = L_1 U_1 = L_2 U_2$

$$\Rightarrow \quad L_2^{-1} L_1 = U_2 U_1^{-1}$$

$L_2^{-1} L_1$ is lower while $U_2 U_1^{-1}$ is upper because the inverse of upper (lower) matrix is again upper (lower) matrix.

$\Rightarrow L_2^{-1} L_1 = D = U_2 U_1^{-1}$, However, $[L_2]_{ii} = 1 = [L_2^{-1}]_{ii}$

$\Rightarrow L_2^{-1} L_1 = I = U_2 U_1^{-1}$ and thus $L_1 = L_2$ and $U_1 = U_2$.

LDU factorization

Note that the upper matrix U can be written as:

$$\begin{pmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & & u_{2n} \\ 0 & 0 & u_{33} & & \vdots \\ & & & \ddots & \\ & & & & u_{nn} \end{pmatrix} = \begin{pmatrix} u_{11} & 0 & \cdots & 0 \\ 0 & u_{22} & 0 \cdots & 0 \\ \vdots & & \ddots & 0 \\ 0 & - & - & u_{nn} \end{pmatrix} \begin{pmatrix} 1 & u_{12}/u_{11} & \cdots & u_{1n}/u_{11} \\ 0 & 1 & & u_{2n}/u_{11} \\ \vdots & & \ddots & \vdots \\ 0 & - & 0 & 1 \end{pmatrix}$$

let $D = \operatorname{diag}(u_{11}, u_{22}, \ldots, u_{nn})$

and redefine U

$$\Rightarrow \quad A = LDU$$

this is called LDU factorization and it is uniquely determined. When A is symmetric, the LDU factorization is

$$A = LDL^T.$$

<u>Note</u>: A symmetric matrix A possessing an LU-factorization in which each pivot is positive is said to be positive definite. In fact

$$A = LDL^T \qquad D = \operatorname{diag}(P_1, -, P_n) \quad P_i > 0$$

set $R = D^{1/2} L^T$ where $D^{1/2} = \operatorname{diag}(\sqrt{P_1}, --, \sqrt{P_n})$

$$\Rightarrow \quad A = L D^{1/2} D^{1/2} L^T$$

$$= \underbrace{R^T R}_{} \qquad R = \text{upper matrix}$$

<u>Cholesky factorization</u>.