Chp. (24.1) Singularities and the Residue Thm.

Def: Isolated Singularity

A complex fn. f has an isolated singularity at z_0 iff f is differentiable in an annulus $0<|z-z_0|<R$, but not at z_0 itself.

Def Classification of Singularities

Let f have an isolated singularity z_0. Let the Laurent expansion of $f(z)$ in an annulus $0<|z-z_0|<R$ be

$$f(z)=\sum_{n=-\infty}^{\infty} c_n(z-z_0)^n .$$

Then

1- z_0 is a removable singularity if $c_n=0$ for $n=-1,-2,\dots$

2. z_0 is a pole of order m $(m>0)$ if $c_m \neq 0$ and $c_{m-1}=c_{m-2}=\dots=0$

3. z_0 is an essential singularity of f if $c_n \neq 0$ for infinitely many positive integers n.

Ex: let $f(z)=\dfrac{1-\cos(z)}{z}$ $\quad 0<|z|<\infty$

we know $\cos(z)=1-\dfrac{z^2}{2!}+\dfrac{z^4}{4!}-\dfrac{z^6}{6!}+\dots \quad \forall z$

$$\Rightarrow f(z)=\frac{1-\cos(z)}{z}=\frac{z}{2!}-\frac{z^3}{4!}+\frac{z^5}{6!}-\dots$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n)!}z^{2n-1} . \quad z\neq 0 .$$

Zero removable singularity of f.

$$g(z)=\begin{cases}(1-\cos(z))/z & z\neq 0\\ 0 & z=0 .\end{cases}$$

Ex: $f(z) = \dfrac{\sin(z)}{z-\pi}$ has a removable singularity at π

$$\sin(z-\pi) = \sin(z)\cos(\pi) - \cos(z)\sin(\pi) = -\sin(z).$$

$$\Rightarrow \sin(z) = -\sin(z-\pi) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}(z-\pi)^{2n+1}.$$

for $z \neq \pi$

$$\frac{\sin(z)}{z-\pi} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}(z-\pi)^{2n} = -1 + \frac{1}{6}(z-\pi)^2 - \frac{1}{120}(z-\pi)^4 + \cdots$$

so define $g(z) = \begin{cases} f(z) & z \neq \pi \\ -1 & z = \pi. \end{cases}$

the extension removes the singularity of f at π, since $f(z) = g(z)$ for $z \neq \pi$ and $g(\pi) = -1$.

~~Def Simple and Double pole~~

Ex: $e^{1/z}$ is defined for all nonzero $z \Rightarrow$

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{z^n} \qquad z \neq 0.$$

the Laurent expansion contains infinitely many negative powers of $z \Rightarrow$ zero is an essential singularity of $e^{1/z}$.

Def: Simple and Double Poles

A pole of order 1 is called a simple pole. A pole of order 2 is called a double pole.

Thm: let f be differentiable in the annulus $0 < |z - z_0| < R$. Then f has a pole of order m at z_0 iff

$$\lim_{z \to z_0} (z-z_0)^m f(z) \text{ exists finite and is nonzero.}$$

Ex: $f(z) = 1/(z+i)$

$$\Rightarrow \lim_{z \to -i} (z+i) f(z) = 1 \neq 0$$

$\Rightarrow$ f has a simple pole at $-i$.

Ex: $g(z) = 1/(z+i)^3$, then

$$\lim_{z \to -i} (z+i)^3 g(z) = \lim_{z \to -i} (z+i)^3 \cdot \frac{1}{(z+i)^3} = 1 \neq 0.$$

so g has a pole of order 3 at $-i$.

Ex: $f(z) = \dfrac{\sin(z)}{z^3}$.

$$\lim_{z \to 0} z^2 \cdot \frac{\sin(z)}{z^3} = \lim_{z \to 0} \frac{\sin(z)}{z} = 1 \neq 0.$$

so f has a double pole at 0.

Thm: Let $f(z) = h(z)/g(z)$, where h and g are differentiable in some open disk about z_0. Suppose $h(z_0) \neq 0$, but g has a zero of order m at z_0. Then f has a pole of order m at z_0.

Ex: $f(z) = \dfrac{1+4z^3}{\sin^6(z)}$ has a pole of order 6 at 0.

Thm: Let $f(z) = h(z)/g(z)$, and suppose h & g are differentiable in some open disk about z_0. Let h have a zero of order k at z_0 and g a zero of order m at z_0 with $m > k$. Then f has a pole of order $m-k$ at z_0.

Ex: Consider $f(z) = \dfrac{(z - 3\pi/2)^4}{\cos^7(z)} = \dfrac{h(z)}{g(z)}$

h has a zero of order 4 at $3\pi/2$, and
g has a zero of order 7 at $3\pi/2$
$\Rightarrow$ f has a pole of order 3 at $3\pi/2$.

Thm (Poles of products)

Let f have a pole of order m at z_0 and let g have a pole of order n at z_0. Then fg has a pole of order m+n at z_0.

Ex: let $f(z) = \dfrac{1}{\cos^4(z)\,(z - \pi/2)^2}$

$$\Rightarrow \quad f(z) = \left[\frac{1}{\cos^4(z)}\right] \times \left[\frac{1}{(z - \pi/2)^2}\right] = g \times h.$$

g has a pole of order 4 at $\pi/2$
h has a pole of order 2 at $\pi/2$
$\Rightarrow$ f has a pole of order 6 at $\pi/2$.