CHARACTERIZATIONS OF BERGMAN SPACE TOEPLITZ OPERATORS WITH HARMONIC SYMBOLS

ISSAM LOUHICHI AND ANDERS OLOFSSON

Abstract. It is well-known that Toeplitz operators on the Hardy space of the unit disc are characterized by the equality $S_1^*TS_1 = T$, where S_1 is the Hardy shift operator. In this paper we give a generalized equality of this type which characterizes Toeplitz operators with harmonic symbols in a class of standard weighted Bergman spaces of the unit disc containing the Hardy space and the unweighted Bergman space. The operators satisfying this equality are also naturally described using a slightly extended form of the Sz.-Nagy-Foias functional calculus for contractions. This leads us to consider Toeplitz operators as integrals of naturally associated positive operator measures in order to take properties of balayage into account.

0. Introduction

Let $n \geq 1$ be an integer. We denote by $A_n(\mathbb{D})$ the Hilbert space of all analytic functions f in the unit disc $\mathbb{D}$ with finite norm

$$\|f\|_{A_n}^2 = \lim_{r\to 1} \int_{\bar{\mathbb{D}}} |f(rz)|^2 d\mu_n(z).$$

The measure $d\mu_1$ is the normalized Lebesgue arc length measure on the unit circle $\mathbb{T}$ and for $n \geq 2$ the measure $d\mu_n$ is the weighted Lebesgue area measure given by

$$d\mu_n(z) = (n-1)(1-|z|^2)^{n-2} dA(z), \quad z \in \mathbb{D},$$

where $dA(z) = dxdy/\pi$, $z = x+iy$, is the planar Lebesgue area measure normalized so that the unit disc $\mathbb{D}$ has area 1. The space $A_1(\mathbb{D})$ is the standard Hardy space, the space $A_2(\mathbb{D})$ is the unweighted Bergman space and in general the space $A_n(\mathbb{D})$ is a so-called standard weighted Bergman space. The norm of $A_n(\mathbb{D})$ is also naturally described by

$$\|f\|_{A_n}^2 = \sum_{k\geq 0} |a_k|^2 \mu_{n;k},$$

where $\mu_{n;k} = 1/\binom{k+n-1}{k}$ for $k \geq 0$, using the power series expansion

$$f(z) = \sum_{k\geq 0} a_k z^k, \quad z \in \mathbb{D}, \tag{0.1}$$

of the function $f \in A_n(\mathbb{D})$ (see [11, Section 1.1]).

Date: March 19, 2007.

1991 *Mathematics Subject Classification.* Primary: 47B35; Secondary: 47A60.

Key words and phrases. Toeplitz operator, standard weighted Bergman space, Sz.-Nagy-Foias functional calculus, Hilbert space of analytic functions, reproducing kernel.

Research of second author supported by the M.E.N.R.T. (France) and the G.S. Magnuson's Fund of the Royal Swedish Academy of Sciences.

The shift operator S_n on the space $A_n(\mathbb{D})$ is the operator defined by

$$(S_nf)(z)=zf(z)=\sum_{k\geq 1}a_{k-1}z^k,\quad z\in\mathbb{D},\tag{0.2}$$

for $f\in A_n(\mathbb{D})$ given by (0.1). Recent work has revealed that the so-called dual shift operator

$$S_n'=S_n(S_n^*S_n)^{-1}$$

plays an important role in the study of shift invariant subspaces of $A_n(\mathbb{D})$. The operator S_n' is a weighted shift operator on $A_n(\mathbb{D})$ which acts as

$$(S_n'f)(z)=\sum_{k\geq 1}\frac{\mu_{n;k-1}}{\mu_{n;k}}a_{k-1}z^k,\quad z\in\mathbb{D},\tag{0.3}$$

on functions $f\in A_n(\mathbb{D})$ given by (0.1).

In recent work [17] on characteristic operator functions we have used the fact that the Bergman shift operator S_n satisfies the operator equality

$$(S_n')^*S_n'=(S_n^*S_n)^{-1}=\sum_{k=0}^{n-1}(-1)^k\binom{n}{k+1}S_n^kS_n^{*k}.\tag{0.4}$$

For full details of proof of formula (0.4) we refer to [17, Section 1].

For $n=1$ equality (0.4) simply says that the Hardy shift operator S_1 is an isometry meaning that $S_1^*S_1=I$. We notice that for $n=2$ equality (0.4) can be written as

$$(S_2')^*S_2'+S_2S_2^*=(S_2^*S_2)^{-1}+S_2S_2^*=2I.\tag{0.5}$$

A similar to (0.5) looking operator inequality has appeared in work of Shimorin [18] on approximation theorems of so-called wandering subspace type and in work of Hedenmalm, Jakobsson and Shimorin [10] on weighted biharmonic Green functions. It is known that the shift operator $S:f\mapsto zf$ in the class of logarithmically subharmonic weighted Bergman spaces in the unit disc with weight functions that are reproducing at the origin satisfies the inequality

$$\|Sf+g\|^2\leq 2(\|f\|^2+\|Sg\|^2)$$

for all functions f and g in the space (see [10, Proposition 6.4]), and that this last inequality is equivalent to the operator inequality

$$(S^*S)^{-1}+SS^*\leq 2I$$

(see [18, Proof of Theorem 3.6]). This last inequality shows a close resemblance to equality (0.5).

We shall study in this paper bounded linear operators $T\in\mathcal{L}(A_n(\mathbb{D}))$ on the Bergman space $A_n(\mathbb{D})$ satisfying the operator equality

$$(S_n')^*TS_n'=\sum_{k=0}^{n-1}(-1)^k\binom{n}{k+1}S_n^kTS_n^{*k}\quad\text{in }\mathcal{L}(A_n(\mathbb{D})).\tag{0.6}$$

We shall show that this equality (0.6) characterizes the Toeplitz operators on $A_n(\mathbb{D})$ with bounded harmonic symbols within the class of all bounded linear operators in $\mathcal{L}(A_n(\mathbb{D}))$ (see Theorem 6.1). By a Toeplitz operator on the Bergman space

$A_n(\mathbb{D})$, $n \geq 2$, with bounded harmonic symbol h in $\mathbb{D}$ we mean the operator T_h in $\mathcal{L}(A_n(\mathbb{D}))$ defined by the formula

$$(T_h f)(z) = \int_{\bar{\mathbb{D}}} \frac{1}{(1-\bar{\zeta}z)^n} h(\zeta) f(\zeta) d\mu_n(\zeta), \quad z \in \mathbb{D}, \tag{0.7}$$

for functions $f \in A_n(\mathbb{D})$. For $n = 1$ we can identify a bounded harmonic function h in $\mathbb{D}$ with its boundary value function in $L^\infty(\mathbb{T})$ by means of the Poisson integral formula (see [12, Lemma III.1.2]). In this way formula (0.7) gives a standard formula for a Hardy space Toeplitz operator. We notice that for $n = 1$ equality (0.6) reduces to $S_1^* T S_1 = T$ and that in this way a well-known characterization of Toeplitz operators on the Hardy space $A_1(\mathbb{D})$ is recovered.

The operators $T \in \mathcal{L}(A_n(\mathbb{D}))$ satisfying equality (0.6) turn out to admit also a simple description using a slightly extended form of the Sz.-Nagy-Foias functional calculus for contractions on Hilbert space taking into account also powers of the adjoint of the operator. Let us describe briefly this functional calculus. Let $T \in \mathcal{L}(\mathcal{H})$ be a contraction on a Hilbert space $\mathcal{H}$ meaning that T is an operator on $\mathcal{H}$ of norm less than or equal to 1. We shall use the notation

$$T(k) = \begin{cases} T^k & \text{for } k \geq 0, \\ T^{*|k|} & \text{for } k < 0, \end{cases} \tag{0.8}$$

which is standard in dilation theory. From the existence of a unitary dilation of T it follows the existence of a positive $\mathcal{L}(\mathcal{H})$-valued operator measure $d\omega_T$ on the unit circle $\mathbb{T}$ such that

$$\int_{\mathbb{T}} e^{ik\theta} d\omega_T(e^{i\theta}) = T(k), \quad k \in \mathbb{Z}.$$

By an approximation argument the operator measure $d\omega_T$ is uniquely determined by this action justifying the notation $d\omega_T$ (see Section 4).

We shall need an appropriate method of summation and as a matter of convenience we shall use the Cesàro summability method. Let $f \in L^1(\mathbb{T})$ be an integrable function on $\mathbb{T}$. The N-th Cesàro mean $\sigma_N f$ of f is defined by the formula

$$(\sigma_N f)(e^{i\theta}) = (K_N * f)(e^{i\theta}) = \sum_{|k| \leq N} \Big(1 - \frac{|k|}{N+1}\Big) \widehat{f}(k) e^{ik\theta}, \quad e^{i\theta} \in \mathbb{T},$$

where $\widehat{f}(k) = \int_{\mathbb{T}} f(e^{i\theta}) e^{-ik\theta} d\theta/2\pi$ is the k-th Fourier coefficient of f and

$$K_N(e^{i\theta}) = \sum_{|k| \leq N} \Big(1 - \frac{|k|}{N+1}\Big) e^{ik\theta}, \quad e^{i\theta} \in \mathbb{T},$$

is the N-th Fejér kernel. It is well-known from harmonic analysis that such means have good approximation properties coming from the fact that $\{K_N\}_{N \geq 1}$ is a well-behaved approximate identity (see [12, Section I.2]).

Let us return to a contraction operator $T \in \mathcal{L}(\mathcal{H})$. It is fairly straightforward to see that if $f \in C(\mathbb{T})$ is a continuous function on $\mathbb{T}$ then

$$\int_{\mathbb{T}} f(e^{i\theta}) d\omega_T(e^{i\theta}) = \lim_{N \to \infty} \sum_{|k| \leq N} \Big(1 - \frac{|k|}{N+1}\Big) \widehat{f}(k) T(k) \quad \text{in } \mathcal{L}(\mathcal{H}) \tag{0.9}$$

with convergence in the uniform operator topology in $\mathcal{L}(\mathcal{H})$ (see Section 4). If the operator measure $d\omega_T$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{T}$ we can more generally integrate essentially bounded measurable functions f

on $\mathbb{T}$ and for such functions $f \in L^\infty(\mathbb{T})$ the above limit (0.9) holds with convergence in the strong operator topology in $\mathcal{L}(\mathcal{H})$ (see Theorem 4.1). It should be mentioned here a result of Sz.-Nagy and Foias related to the structure of the minimal unitary dilation which gives that the operator measure $d\omega_T$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{T}$ if $T \in \mathcal{L}(\mathcal{H})$ is a completely non-unitary contraction (see [19, Theorem II.6.4]).

Let us return to a bounded linear operator $T \in \mathcal{L}(A_n(\mathbb{D}))$ on the Bergman space $A_n(\mathbb{D})$. We shall show that such an operator T satisfies equality (0.6) if and only if it has the form of a functional calculus integral

$$T = \int_{\mathbb{T}} f(e^{i\theta}) d\omega_{S_n}(e^{i\theta}) \quad \text{in } \mathcal{L}(A_n(\mathbb{D}))$$

of a function $f \in L^\infty(\mathbb{T})$ relative to the Bergman shift operator S_n, and that the norm equality $\|T\| = \|f\|_\infty$ holds (see Theorem 5.1).

We discuss also these operator integrals arising from the shift operator

$$S : f \mapsto zf, \quad f \in \mathcal{H},$$

on a Hilbert space $\mathcal{H}$ of analytic functions on the unit disc $\mathbb{D}$ such that the operator S is a contraction on $\mathcal{H}$. For such an operator S the operator measure $d\omega_S$ is always absolutely continuous with respect to Lebesgue measure on $\mathbb{T}$ and we have the norm equality

$$\| \int_{\mathbb{T}} f(e^{i\theta}) d\omega_S(e^{i\theta}) \| = \|f\|_\infty, \quad f \in L^\infty(\mathbb{T})$$

(see Theorem 5.2). We mention that this last norm equality also follows by a result of Conway and Ptak [9, Theorem 2.2] using that the $H^\infty(\mathbb{D})$-functional calculus for the shift operator S is isometric. We also show that the operator

$$\int_{\mathbb{T}} f(e^{i\theta}) d\omega_S(e^{i\theta}) \quad \text{in } \mathcal{L}(\mathcal{H})$$

is compact if and only if $f = 0$ (see Theorem 5.3). These two results generalize to the context of Hilbert spaces of analytic functions well-known properties of Toeplitz operators on the Hardy space of the unit disc.

Above we have described two seemingly different looking characterizations of the class of operators $T \in \mathcal{L}(A_n(\mathbb{D}))$ satisfying equality (0.6), namely, first as Toeplitz operators T_h on $A_n(\mathbb{D})$ with symbols h that are bounded harmonic functions in $\mathbb{D}$, and then as functional calculus integrals of functions $f \in L^\infty(\mathbb{T})$ with respect the shift operator S_n. The correspondence between these two classes of operators is given by

$$T_h = \int_{\mathbb{T}} f(e^{i\theta}) d\omega_{S_n}(e^{i\theta}) \quad \text{in } \mathcal{L}(A_n(\mathbb{D})),$$

where $h = P[f]$ is the Poisson integral of $f \in L^\infty(\mathbb{T})$. We interpret this equality that some sort of balayage is going on and that Toeplitz operators should be considered differently as integrals of naturally associated positive operator measures in order to make this process of balayage more transparent. We discuss some rudiments of such a theory in Section 7 in this paper.

Let us describe briefly how we analyze the operators satisfying equality (0.6). The function space $A_n(\mathbb{D})$ is equipped with a natural group of translations $\tau_{e^{i\theta}}$ parametrized by elements on the unit circle $e^{i\theta} \in \mathbb{T}$. It is a general fact that a bounded linear operator on such a space can be decomposed into homogeneous parts with respect to such a group of translations and the operator can then be reconstructed from its homogeneous parts as a limit in the strong operator topology using an appropriate method of summation analogous to what is usually done in harmonic analysis. We discuss this construction in Section 2. By homogeneity type of arguments an operator satisfying (0.6) can be described as an operator with all of its homogeneous parts satisfying (0.6), and for an operator of a fixed degree of homogeneity we can apply shifts or backward shifts in order to reduce to the case of an operator which is homogeneous of degree 0 which is nothing else but a Fourier multiplier. Arguing this way we show that an operator $T \in \mathcal{L}(A_n(\mathbb{D}))$ satisfies equality (0.6) if and only if every k-th homogeneous part T_k of T is a constant multiple of $S_n(k)$ (see Theorem 3.1). We then use this analysis to arrive at the descriptions indicated above.

Partially driven by classical work of Brown and Halmos [7] on algebraic properties of Toeplitz operators on the Hardy space much effort has been put into the study of the corresponding questions for Toeplitz operators on the unweighted Bergman space $A_2(\mathbb{D})$: Axler and Čučković [6] have solved the commutativity problem of when two Toeplitz operators with harmonic symbols commute. Ahern and Čučković [2] have solved the product problem of when the product of two Toeplitz operators with harmonic symbols is again a Toeplitz operator with harmonic symbol; see also Ahern [1]. From this point of view we give in this paper the generalization of Theorem 6 of Brown and Halmos [7] to the Bergman spaces $A_n(\mathbb{D})$.

The authors thank Elizabeth Strouse for useful discussions.

1. Preliminaries

We collect in this section some constructions and formulas involving the shift operator S_n on $A_n(\mathbb{D})$ in order to make the presentation in later sections more efficient. We also sketch some background on positive operator measures.

Shifts and backward shifts on the Bergman space. The adjoint shift operator S_n^* in $\mathcal{L}(A_n(\mathbb{D}))$ acts as

$$(S_n^* f)(z) = \sum_{k\geq 0} \frac{\mu_{n;k+1}}{\mu_{n;k}} a_{k+1} z^k, \quad z \in \mathbb{D}, \tag{1.1}$$

on functions $f \in A_n(\mathbb{D})$ given by (0.1). Notice that the space

$$\ker S_n^* = A_n(\mathbb{D}) \ominus S_n(A_n(\mathbb{D}))$$

consists of the constant functions in $\mathbb{D}$. The operator $L_n = (S_n^* S_n)^{-1} S_n^*$ in $\mathcal{L}(A_n(\mathbb{D}))$ is the left-inverse of S_n with kernel $\ker L_n = \ker S_n^*$. The operator L_n acts as

$$(L_n f)(z) = \frac{f(z) - f(0)}{z} = \sum_{k\geq 0} a_{k+1} z^k, \quad z \in \mathbb{D},$$

on functions $f \in A_n(\mathbb{D})$ given by (0.1). In other words, the operator L_n is the backward shift operator on $A_n(\mathbb{D})$.

As is apparent from the introduction we shall make use of the dual shift operator

$$S_n' = S_n (S_n^* S_n)^{-1} \quad \text{in } \mathcal{L}(A_n(\mathbb{D}))$$

which is a weighted shift operator on $A_n(\mathbb{D})$ whose action is given by (0.3). Notice that $(S_n')^* = (S_n^* S_n)^{-1} S_n^* = L_n$ in $\mathcal{L}(A_n(\mathbb{D}))$ and that the operator

$$L_n' = ((S_n')^* S_n')^{-1} (S_n')^* = S_n^* \quad \text{in } \mathcal{L}(A_n(\mathbb{D}))$$

is the left-inverse of S_n' with kernel $\ker L_n' = \ker S_n^*$ consisting of the constant functions.

Positive operator measures. Let $(X, \mathfrak{S})$ be a measure space consisting of a set X and a σ-algebra $\mathfrak{S}$ of measurable subsets of X, and let $\mathcal{H}$ a Hilbert space. By a positive $\mathcal{L}(\mathcal{H})$-valued operator measure $d\mu$ we mean a finitely additive set function $\mu : \mathfrak{S} \to \mathcal{L}(\mathcal{H})$ such that $\mu(S)$ is a positive operator in $\mathcal{L}(\mathcal{H})$ for every $S \in \mathfrak{S}$ and the marginal set functions $\mu_{x,y}$, $x, y \in \mathcal{H}$, defined by

$$\mu_{x,y}(S) = \langle \mu(S)x, y\rangle, \quad S \in \mathfrak{S},$$

are all complex measures in the usual sense. Notice that this amounts to saying that $d\mu_{x,x}$ is a finite positive measure for every $x \in \mathcal{H}$. If f is a measurable complex-valued function such that f is integrable with respect to $d\mu_{x,y}$ for all $x, y \in \mathcal{H}$ we say that f is integrable with respect to $d\mu$ and we define the integral $\int_X f d\mu$ as an operator in $\mathcal{L}(\mathcal{H})$ by the requirement that

$$\langle \int_X f(s) d\mu(s) x, y\rangle = \int_X f(s) d\mu_{x,y}(s), \quad x, y \in \mathcal{H}.$$

It is straightforward to see that every bounded measurable function f on X is integrable with respect to $d\mu$ and that we have the norm bound

$$\| \int_X f(s) d\mu(s) \| \leq \|\mu(X)\| \|f\|_\infty, \tag{1.2}$$

where

$$\|f\|_\infty = \inf \{ c > 0 : \ \mu(\{x \in X : \ |f(x)| > c\}) = 0 \quad \text{in } \mathcal{L}(\mathcal{H}) \}$$

is the essential supremum of f (see [15, Section 1]).

A positive operator measure $d\mu$ with $\mu(X) = I$ such that $\mu(S)$ is an orthogonal projection in $\mathcal{L}(\mathcal{H})$ for every $S \in \mathfrak{S}$ is called a *spectral* measure. We record also that a measurable function f is integrable with respect to a spectral measure $d\mu = dE$ if and only if it is essentially bounded and that equality always holds in (1.2) in this case. Positive operator measures $d\mu$ such that $\mu(X) = I$ are in the literature sometimes called quasi or semi spectral measures.

2. Decomposition of an operator into homogeneous parts

We shall discuss in this section a decomposition of an operator into homogeneous parts with respect to translations. This decomposition is of somewhat independent interest and we discuss it here in some more generality than needed for our applications.

Let $\mathcal{X}$ be a Banach space and denote by $\mathcal{L}(\mathcal{X})$ the space of bounded linear operators on $\mathcal{X}$. We assume that the circle group $\mathbb{T}$ operates on elements in $\mathcal{X}$ in such a way that the map

$$\tau : \mathbb{T} \ni e^{i\theta} \mapsto \tau_{e^{i\theta}} \in \mathcal{L}(\mathcal{X})$$

is continuous in the strong operator topology in $\mathcal{L}(\mathcal{X})$, the operator τ_1 is the identity operator I in $\mathcal{L}(\mathcal{X})$ and

$$\tau_{e^{i(\theta_1+\theta_2)}} = \tau_{e^{i\theta_1}} \tau_{e^{i\theta_2}} \quad \text{in } \mathcal{L}(\mathcal{X}), \quad e^{i\theta_1}, e^{i\theta_2} \in \mathbb{T}.$$

Notice that this group structure simplifies the continuity requirement of the map τ to the requirement that

$$\lim_{e^{i\theta}\to 1} \tau_{e^{i\theta}} x = x \quad \text{in } \mathcal{X}, \quad x \in \mathcal{X},$$

with convergence in the norm of $\mathcal{X}$. Notice also that the operator norms $\|\tau_{e^{i\theta}}\|$ are uniformly bounded by the Banach-Steinhaus theorem. Let $T \in \mathcal{L}(\mathcal{X})$ and consider the operators

$$T_k x = \frac{1}{2\pi} \int_{\mathbb{T}} e^{-ik\theta} \tau_{e^{-i\theta}} T \tau_{e^{i\theta}} x \, d\theta, \quad x \in \mathcal{X}, \tag{2.1}$$

for $k \in \mathbb{Z}$. The integral in (2.1) is interpreted as an $\mathcal{X}$-valued integral of a continuous function. Clearly T_k is an operator in $\mathcal{L}(\mathcal{X})$ and the operator norm of T_k is bounded by a constant times the norm of T. A change of variables shows that the operator T_k has the homogeneity property that

$$\tau_{e^{-i\theta}} T_k \tau_{e^{i\theta}} = e^{ik\theta} T_k \quad \text{in } \mathcal{L}(\mathcal{X}), \quad e^{i\theta} \in \mathbb{T}.$$

We call such an operator *homogeneous* of degree k with respect to translations. It is straightforward to see that if $T_j \in \mathcal{L}(\mathcal{X})$ is homogeneous of degree j and $T_k \in \mathcal{L}(\mathcal{X})$ is homogeneous of degree k, then the product $T_j T_k \in \mathcal{L}(\mathcal{X})$ is homogeneous of degree $j+k$. We notice also that if $\mathcal{X} = \mathcal{H}$ is a Hilbert space and $T \in \mathcal{L}(\mathcal{H})$ is homogeneous of degree k, then the adjoint operator $T^* \in \mathcal{L}(\mathcal{H})$ is homogeneous of degree $-k$.

Let us return to the above general discussion. Keeping in mind that for $x \in \mathcal{X}$ the map

$$\mathbb{T} \ni e^{i\theta} \mapsto \tau_{e^{-i\theta}} T \tau_{e^{i\theta}} x \in \mathcal{X},$$

is a continuous $\mathcal{X}$-valued function on $\mathbb{T}$ we have by a standard argument that

$$Tx = \lim_{N\to\infty} \frac{1}{2\pi} \int_{\mathbb{T}} K_N(e^{i\theta}) \tau_{e^{-i\theta}} T \tau_{e^{i\theta}} x \, d\theta \quad \text{in } \mathcal{X}, \quad x \in \mathcal{X},$$

whenever $\{K_N\}_{N\geq 1}$ is a suitable approximate identity (see [12, Section I.2]). In particular, choosing K_N as the N-th Fejér kernel we obtain that

$$Tx = \lim_{N\to\infty} \sum_{|k|\leq N} \Big(1 - \frac{|k|}{N+1}\Big) T_k x \quad \text{in } \mathcal{X}, \quad x \in \mathcal{X},$$

meaning that the operator T can be reconstructed from its homogeneous parts T_k by means of Cesàro summation in the strong operator topology in $\mathcal{L}(\mathcal{X})$.

We shall apply the above discussion when $\mathcal{X}$ is the Bergman space $A_n(\mathbb{D})$ and the translations $\tau_{e^{i\theta}}$ are acting as

$$(\tau_{e^{i\theta}} f)(z) = f(e^{-i\theta} z), \quad z \in \mathbb{D},$$

on functions $f \in A_n(\mathbb{D})$ as is customary in harmonic analysis. Notice that here the translations $\tau_{e^{i\theta}}$ are unitary operators in $\mathcal{L}(A_n(\mathbb{D}))$ and that this gives that

$$\tau_{e^{i\theta}}^* = \tau_{e^{-i\theta}} \quad \text{in } \mathcal{L}(A_n(\mathbb{D})), \quad e^{i\theta} \in \mathbb{T}.$$

It is straightforward to see that an operator $T \in \mathcal{L}(A_n(\mathbb{D}))$ is homogeneous of degree 0 if and only if it acts like a Fourier multiplier in the sense that

$$(Tf)(z) = \sum_{k\geq 0} t_k a_k z^k, \quad z \in \mathbb{D},$$

for $f \in A_n(\mathbb{D})$ given by (0.1). Boundedness of the operator T corresponds to the multiplier sequence $\{t_k\}_{k\geq 0}$ being bounded. The Bergman shift operator S_n is homogeneous of degree 1, and such homogeneity has also the dual shift operator S_n'. The left-inverses L_n and L_n' are homogeneous of degree -1.

We can think of an operator T in $\mathcal{L}(A_n(\mathbb{D}))$ as given by an infinite matrix $\{t_{jk}\}_{j,k\geq 0}$ relative to the orthogonal basis $\{e_k\}_{k\geq 0}$ of monomials

$$e_k(z) = z^k, \quad z \in \mathbb{D},$$

for $k \geq 0$. The action of the operator T is then described by

$$T(e_k) = \sum_{j\geq 0} t_{jk} e_j \quad \text{in } A_n(\mathbb{D}), \quad \langle Te_k, e_j\rangle_{A_n} = t_{jk}\mu_{n;j}.$$

In this context the m-th homogeneous part T_m of T corresponds to the matrix $\{\delta_{m,j-k}t_{jk}\}_{j,k\geq 0}$ obtained from the matrix $\{t_{jk}\}_{j,k\geq 0}$ by putting all elements outside the diagonal $m = j - k$ equal to 0.

3. Powers of shifts and adjoint shifts

Recall the notation (0.8). We shall prove in this section that an operator $T \in \mathcal{L}(A_n(\mathbb{D}))$ satisfies equality (0.6) if and only if every k-th homogeneous part T_k of T is a constant multiple of $S_n(k)$ (see Theorem 3.1). The proof will proceed in several steps.

We first show that powers of shifts and adjoint shifts satisfy equality (0.6).

Proposition 3.1. *The operators $S_n(k)$ in $\mathcal{L}(A_n(\mathbb{D}))$ for $k \in \mathbb{Z}$ all satisfy equality (0.6).*

Proof. We first show that S_n^k satisfies (0.6) for $k \geq 0$. We have that

$$(S_n')^* S_n^k S_n' = (S_n^*S_n)^{-1} S_n^* S_n^k S_n (S_n^*S_n)^{-1} = S_n^k (S_n^*S_n)^{-1}.$$

By equality (0.4) we conclude that

$$(S_n')^* S_n^k S_n' = S_n^k \Big(\sum_{j=0}^{n-1} (-1)^j \binom{n}{j+1} S_n^j S_n^{*j} \Big) = \sum_{j=0}^{n-1} (-1)^j \binom{n}{k+1} S_n^j S_n^k S_n^{*j}.$$

This shows that S_n^k satisfies (0.6).

By a passage to adjoints we see that also the operators S_n^{*k} for $k \geq 0$ satisfy equality (0.6). This completes the proof of the proposition. □

We next consider a Fourier multiplier in $\mathcal{L}(A_n(\mathbb{D}))$ satisfying equality (0.6).

Lemma 3.1. *Let $T \in \mathcal{L}(A_n(\mathbb{D}))$ be homogeneous of degree 0, and assume that T satisfies equality (0.6). Then the operator T is a constant multiple of the identity operator I.*

Proof. As we have pointed out in Section 2 the assumption of homogeneity of degree 0 means that the operator $T \in \mathcal{L}(A_n(\mathbb{D}))$ acts as

$$(Tf)(z) = \sum_{j\geq 0} t_j a_j z^j, \quad z \in \mathbb{D},$$

on functions $f \in A_n(\mathbb{D})$ given by (0.1) for some bounded sequence $\{t_j\}_{j\geq 0}$ of complex numbers.

Let us prove that $t_j = t_0$ for $j \geq 0$. The assumption that T satisfies equality (0.6) means that

$$\langle TS_n' f, S_n' f\rangle_{A_n} = \sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} \langle TS_n^{*k} f, S_n^{*k} f\rangle_{A_n}, \quad f \in A_n(\mathbb{D}). \tag{3.1}$$

Let $f \in A_n(\mathbb{D})$ be a function of the form (0.1). By (0.3) we have that

$$\langle TS_n' f, S_n' f\rangle_{A_n} = \sum_{j\geq 1} t_j \frac{\mu_{n;j-1}^2}{\mu_{n;j}} |a_{j-1}|^2 = \sum_{j \geq 0} t_{j+1} \frac{\mu_{n;j}^2}{\mu_{n;j+1}} |a_j|^2. \tag{3.2}$$

By (1.1) we have that

$$(S_n^{*k} f)(z) = \sum_{j\geq 0} \frac{\mu_{n;j+k}}{\mu_{n;j}} a_{j+k} z^j, \quad z \in \mathbb{D},$$

for $k \geq 0$. Computing scalar products and summing we now have that

$$\begin{aligned} \sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} \langle TS_n^{*k} f, S_n^{*k} f\rangle_{A_n} &= \sum_{k=0}^{n-1} \sum_{j\geq 0} (-1)^k \binom{n}{k+1} t_j \frac{\mu_{n;j+k}^2}{\mu_{n;j}} |a_{j+k}|^2 \\ &= \sum_{j\geq 0} \Big(\sum_{k=0}^{\min(j,n-1)} (-1)^k \binom{n}{k+1} \frac{t_{j-k}}{\mu_{n;j-k}} \Big) |a_j|^2 \mu_{n;j}^2, \end{aligned} \tag{3.3}$$

where the last equality follows by a change of order of summation. Varying the function $f \in A_n(\mathbb{D})$ of the form (0.1) the above equalities (3.1), (3.2) and (3.3) give that

$$\frac{t_{j+1}}{\mu_{n;j+1}} = \sum_{k=0}^{\min(j,n-1)} (-1)^k \binom{n}{k+1} \frac{t_{j-k}}{\mu_{n;j-k}} \tag{3.4}$$

for $j \geq 0$. We introduce now the function

$$f(z) = \sum_{j\geq 0} \frac{t_j}{\mu_{n;j}} z^j, \quad z \in \mathbb{D}.$$

Notice that the sum in (3.4) is equal to the j-th coefficient in the power series expansion of the function

$$\Big(\sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} z^k \Big) f(z), \quad z \in \mathbb{D}.$$

Multiplying (3.4) by z^j and summing for $j \geq 0$ we have that

$$\frac{f(z) - t_0}{z} = \Big(\sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} z^k \Big) f(z) = \frac{1-(1-z)^n}{z} f(z), \quad z \in \mathbb{D}.$$

We can now solve this last equation for $f(z)$ to conclude that

$$f(z) = t_0 \frac{1}{(1-z)^n} = t_0 \sum_{j\geq 0} \frac{1}{\mu_{n;j}} z^j, \quad z \in \mathbb{D}.$$

This shows that $t_j = t_0$ for all $j \geq 0$. This completes the proof of the lemma. $\square$

We shall need also the following lemma concerned with operators $T \in \mathcal{L}(A_n(\mathbb{D}))$ of positive degree of homogeneity.

Lemma 3.2. *Let $T \in \mathcal{L}(A_n(\mathbb{D}))$ be homogeneous of degree $m \geq 0$. Then the range of T is contained in the range of S_n^m.*

Proof. Since the operator T is homogeneous of degree m it must equal its m-th homogeneous part T_m by considerations from Section 2. Recall also the matrix representation of the m-th homogeneous part from Section 2 saying that the operator $T = T_m$ acts as

$$Te_k = t_{k+m,k}e_{k+m}, \quad k \geq 0,$$

relative to the orthogonal basis of monomials $e_k(z) = z^k$. It is easy to see that the operator S_n^m has closed range. We conclude that the range of T is contained in the range of S_n^m. This completes the proof of the lemma. □

We can now describe a general operator $T \in \mathcal{L}(A_n(\mathbb{D}))$ satisfying equality (0.6).

Theorem 3.1. *Let $T \in \mathcal{L}(A_n(\mathbb{D}))$. Then the operator T satisfies equality (0.6) if and only if every k-th homogeneous part T_k of T is a constant multiple of $S_n(k)$.*

Proof. The if-part is evident by Proposition 3.1 and summability considerations from Section 2.

Let now $T \in \mathcal{L}(A_n(\mathbb{D}))$ be a general operator satisfying (0.6). We first show that every k-th homogeneous part T_k of T satisfies (0.6). Recall that the operators S_n and S_n' are both homogeneous of degree 1. Using (0.6) and these homogeneity properties we compute that

$$\begin{aligned}(S_n')^*\tau_{e^{-i\theta}}T\tau_{e^{i\theta}}S_n' = \tau_{e^{-i\theta}}(S_n')^*TS_n'\tau_{e^{i\theta}} &= \tau_{e^{-i\theta}}\Big(\sum_{k=0}^{n-1}(-1)^k\binom{n}{k+1}S_n^kTS_n^{*k}\Big)\tau_{e^{i\theta}} \\ &= \sum_{k=0}^{n-1}(-1)^k\binom{n}{k+1}S_n^k\tau_{e^{-i\theta}}T\tau_{e^{i\theta}}S_n^{*k} \quad \text{in } \mathcal{L}(A_n(\mathbb{D})).\end{aligned}$$

We can now pass to the k-th homogeneous part using (2.1) to conclude that T_k satisfies (0.6) for every $k \in \mathbb{Z}$.

We shall next show that if an operator $T_k \in \mathcal{L}(A_n(\mathbb{D}))$ is homogeneous of degree $k \geq 0$ and satisfies (0.6), then T_k is a constant multiple of S_n^k. Recall that $(S_n')^* = L_n$. We first show that $L_n^kT_k$ satisfies (0.6). Since T_k satisfies (0.6) we have that

$$(S_n')^*L_n^kT_kS_n' = L_n^k(S_n')^*T_kS_n' = L_n^k\Big(\sum_{j=0}^{n-1}(-1)^j\binom{n}{j+1}S_n^jT_kS_n^{*j}\Big). \tag{3.5}$$

By Lemma 3.2 the range of T_k is contained in the range of S_n^k. This gives that the operator T_k factorizes as $T_k = S_n^kL_n^kT_k$. By (3.5) we conclude that $L_n^kT_k$ satisfies (0.6). The operator $L_n^kT_k$ is also homogeneous of degree 0, and Lemma 3.1 applies to give that $L_n^kT_k = cI$. We conclude that $T_k = S_n^kL_n^kT_k = cS_n^k$.

We now finish the proof of the theorem. We know that each homogeneous part T_k of T satisfies (0.6). By the result of the previous paragraph we have that T_k is a constant multiple of S_n^k if $k \geq 0$. Assume next that $k < 0$. Then T_k^* is homogeneous of degree $|k|$ and by a passage to adjoints we see that also T_k^* satisfies (0.6). The result of the previous paragraph applies to give that T_k^* is a constant multiple of $S_n^{|k|}$. Taking adjoints again we see that T_k is a constant multiple of $S_n^{*|k|} = S_n(k)$. This completes the proof of the theorem. □

4. Functional calculus for contractions

In this section we shall discuss an extension of the Sz.-Nagy-Foias functional calculus for contraction operators on Hilbert space taking into account also powers of the adjoint of the operator. Let us proceed to describe this functional calculus.

Let $T \in \mathcal{L}(\mathcal{H})$ be a contraction operator on a Hilbert space $\mathcal{H}$ meaning that $\|T\| \leq 1$. A classical result of Sz.-Nagy concerns the existence of a unitary dilation of T, that is, a unitary operator $U \in \mathcal{L}(\mathcal{K})$ on a larger Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ as a closed subspace such that

$$T^k = P_{\mathcal{H}} U^k|_{\mathcal{H}} \quad \text{in } \mathcal{L}(\mathcal{H})$$

for $k \geq 0$, where $P_{\mathcal{H}}$ denotes the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$ (see [19, Chapter I]). A related construction is that of a positive $\mathcal{L}(\mathcal{H})$-valued operator measure $d\omega_T$ on the unit circle $\mathbb{T}$ having Fourier coefficients

$$\widehat{\omega}_T(k) = \int_{\mathbb{T}} e^{-ik\theta} d\omega_T(e^{i\theta}) = \begin{cases} T^{*k} & \text{for } k \geq 0, \\ T^{|k|} & \text{for } k < 0. \end{cases}$$

Notice that $d\omega_T$ is uniquely determined by its Fourier coefficients. This operator measure $d\omega_T$ can be obtained as the compression to $\mathcal{H}$ of the spectral measure for a unitary dilation $U \in \mathcal{L}(\mathcal{K})$ of T. Alternatively, the operator measure $d\omega_T$ can be constructed using operator-valued Poisson integrals (see [15] for a construction in the context of the n-torus $\mathbb{T}^n$).

As indicated in Section 1 integration with respect to $d\omega_T$ is a continuous linear map

$$C(\mathbb{T}) \ni f \mapsto \int_{\mathbb{T}} f(e^{i\theta}) d\omega_T(e^{i\theta}) \in \mathcal{L}(\mathcal{H})$$

of $C(\mathbb{T})$ into $\mathcal{L}(\mathcal{H})$ of norm equal to 1. Let $f \in C(\mathbb{T})$ and consider the Cesàro means $\sigma_N f = K_N * f$ of f defined as in the introduction. It is well-known that the Cesàro means $\sigma_N f$ converges to f in $C(\mathbb{T})$ (see [12, Section I.2]). Integrating with respect to $d\omega_T$ we arrive at the limit assertion that

$$\int_{\mathbb{T}} f(e^{i\theta}) d\omega_T(e^{i\theta}) = \lim_{N\to\infty} \sum_{|k|\leq N} \Big(1 - \frac{|k|}{N+1}\Big) \widehat{f}(k) T(k) \quad \text{in } \mathcal{L}(\mathcal{H})$$

with convergence in the uniform operator topology in $\mathcal{L}(\mathcal{H})$, where $\widehat{f}(k)$ denotes the k-th Fourier coefficient of f.

If the operator measure $d\omega_T$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{T}$ we can more generally integrate functions in $L^\infty(\mathbb{T})$. We shall next study the corresponding approximation property of such integrals. Here should be mentioned a result of Sz.-Nagy and Foias which asserts that the spectral measure of the minimal unitary dilation of a completely non-unitary contraction is absolutely continuous with respect to Lebesgue measure on $\mathbb{T}$ (see [19, Theorem II.6.4]). This result gives by a compression argument that the operator measure $d\omega_T$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{T}$ if $T \in \mathcal{L}(\mathcal{H})$ is a completely non-unitary contraction.

We shall need the following lemma from integration theory.

Lemma 4.1. *Let $d\mu$ be a positive $\mathcal{L}(\mathcal{H})$-valued operator measure on a measure space X such that $\mu(X) = I$, and let f be a measurable function on X which is* *square integrable with respect to $d\mu$. Then*

$$\|\int_X f(s)d\mu(s)x\|^2 \leq \int_X |f(s)|^2 d\mu_{x,x}(s), \quad x \in \mathcal{H}. \tag{4.1}$$

If $d\mu = dE$ is a spectral measure, then equality holds in (4.1).

Sketch of proof. We consider first the case when $d\mu = dE$ is a spectral measure. For f a simple function the assertion of equality in (4.1) can be verified by straightforward computation. The case of a general function $f \in L^\infty(dE)$ then follows by approximation by simple functions.

Let now $d\mu$ be a positive operator measure such that $\mu(X) = I$ in $\mathcal{L}(\mathcal{H})$. By a result of Neumark [14] the operator measure $d\mu$ can be dilated to a spectral measure dE (see formula (7.1) in Section 7). Using this dilation we can verify inequality (4.1) for f bounded. The case of a general function f follows by an approximation argument. □

We can now prove the corresponding approximation property for integrals of $L^\infty(\mathbb{T})$-functions.

Theorem 4.1. *Let $T \in \mathcal{L}(\mathcal{H})$ be a contraction operator such that the operator measure $d\omega_T$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{T}$, and let $f \in L^\infty(\mathbb{T})$. Then*

$$\int_{\mathbb{T}} f(e^{i\theta}) d\omega_T(e^{i\theta}) = \lim_{N\to\infty} \sum_{|k|\leq N} \Big(1 - \frac{|k|}{N+1}\Big) \widehat{f}(k) T(k) \quad \textit{in } \mathcal{L}(\mathcal{H})$$

with convergence in the strong operator topology in $\mathcal{L}(\mathcal{H})$.

Proof. Let K_N be the N-th Fejér kernel and denote by $\sigma_N f = K_N * f$ the N-th Cesàro mean of $f \in L^\infty(\mathbb{T})$. It is well-known that $\lim_{N\to\infty} \sigma_N f = f$ pointwise a.e. on $\mathbb{T}$ and that $\|\sigma_N f\|_\infty \leq \|f\|_\infty$ for all $N \geq 1$ (see [12, Section I.3]). Let $x \in \mathcal{H}$. By Lemma 4.1 we have that

$$\|\int_{\mathbb{T}} \big(f(e^{i\theta}) - (\sigma_N f)(e^{i\theta})\big) d\omega_T(e^{i\theta}) x\|^2 \leq \int_{\mathbb{T}} \big|f(e^{i\theta}) - (\sigma_N f)(e^{i\theta})\big|^2 d(\omega_T)_{x,x}(e^{i\theta}).$$

By dominated convergence the integral on the right-hand side tends to 0 as $N \to \infty$. We conclude that

$$\int_{\mathbb{T}} f(e^{i\theta}) d\omega_T(e^{i\theta}) = \lim_{N\to\infty} \int_{\mathbb{T}} (\sigma_N f)(e^{i\theta}) d\omega_T(e^{i\theta}) \quad \text{in } \mathcal{L}(\mathcal{H})$$

with convergence in the strong operator topology in $\mathcal{L}(\mathcal{H})$. A computation shows that

$$\int_{\mathbb{T}} (\sigma_N f)(e^{i\theta}) d\omega_T(e^{i\theta}) = \sum_{|k|\leq N} \Big(1 - \frac{|k|}{N+1}\Big) \widehat{f}(k) T(k) \quad \text{in } \mathcal{L}(\mathcal{H}).$$

This completes the proof of the theorem. □

Remark 4.1. We remark that the argument in the proof of Theorem 4.1 gives that

$$\lim_{k\to\infty} \int_{\mathbb{T}} f_k(e^{i\theta}) d\omega_T(e^{i\theta}) = \int_{\mathbb{T}} f_0(e^{i\theta}) d\omega_T(e^{i\theta}) \quad \text{in } \mathcal{L}(\mathcal{H}) \tag{4.2}$$

with convergence in the strong operator topology in $\mathcal{L}(\mathcal{H})$ whenever $\{f_k\}$ is a bounded sequence of functions in $L^\infty(\mathbb{T})$ such that $f_k(e^{i\theta}) \to f_0(e^{i\theta})$ for a.e. $e^{i\theta} \in \mathbb{T}$ or $f_k \to f_0$ in measure. This can be compared with the limit assertion that (4.2) holds in the weak operator topology in $\mathcal{L}(\mathcal{H})$ whenever $f_k \to f_0$ in the weak* topology of $L^\infty(\mathbb{T})$ which evident from the assumption that $d\omega_T$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{T}$. By the weak* topology on $L^\infty(\mathbb{T})$ we mean the topology on $L^\infty(\mathbb{T})$ which is induced from $L^\infty(\mathbb{T})$ being the dual of $L^1(\mathbb{T})$.

We mention that the Sz.-Nagy-Foias functional calculus for contractions concerns the corresponding Abel summability statements for the calculus

$$u(T) = \lim_{r\to 1} \sum_{k\geq 0} a_k r^k T^k \quad \text{in } \mathcal{L}(\mathcal{H}),$$

where u is a bounded analytic function in $\mathbb{D}$ with Taylor coefficients $\{a_k\}_{k\geq 0}$ (see [19, Chapter III]). Our functional calculus gives this calculus as a special case when applied to functions $f \in L^\infty(\mathbb{T})$ with vanishing negative Fourier coefficients.

The arguments in this section follow those of Sz.-Nagy and Foias [19, Section III.2] and are repeated here for the matter of convenience.

5. Functional calculus for shift operators

In this section we shall describe the operators $T \in \mathcal{L}(A_n(\mathbb{D}))$ satisfying equality (0.6) using the functional calculus from Section 4 for the Bergman shift operator S_n on $A_n(\mathbb{D})$. We also discuss the functional calculus from Section 4 when applied to the shift operator on a Hilbert space of analytic functions on the unit disc.

We recall that the operator measure $d\omega_T$ from Section 4 is absolutely continuous with respect to Lebesgue measure on $\mathbb{T}$ if T is a completely non-unitary contraction (see [19, Theorem II.6.4]). For a contraction $T \in \mathcal{L}(\mathcal{H})$ such that $\lim_{k\to\infty} T^k = 0$ in the strong operator topology in $\mathcal{L}(\mathcal{H})$ this assertion of absolute continuity can be seen more easily since such an operator T can always be modeled as part of a vector-valued adjoint Hardy shift S_1^* (see [19, Subsection I.10.1] or [16] for constructions). See also Proposition 7.2 in Section 7.

Let $\mathcal{H}$ be a Hilbert space of analytic functions on the unit disc $\mathbb{D}$. By this we mean that the elements in $\mathcal{H}$ are analytic functions in $\mathbb{D}$ and that the point evaluations at points in $\mathbb{D}$ are continuous linear functionals on $\mathcal{H}$. Associated to such a space $\mathcal{H}$ we have a reproducing kernel function $K_\mathcal{H}$ which is the function $K_\mathcal{H} : \mathbb{D}\times\mathbb{D} \to \mathbb{C}$ uniquely determined by the properties that $K_\mathcal{H}(\cdot,\zeta) \in \mathcal{H}$ for every $\zeta \in \mathbb{D}$ and

$$f(\zeta) = \langle f, K_\mathcal{H}(\cdot,\zeta)\rangle, \quad \zeta \in \mathbb{D},$$

for every function $f \in \mathcal{H}$. This last property is called the reproducing property of the kernel function $K_\mathcal{H}$ (see [5, Section I.2]).

We shall need the following lemma.

Lemma 5.1. *Let $\mathcal{H}$ be a Hilbert space of analytic functions on $\mathbb{D}$ such that the shift operator $S : f \mapsto zf$ acts as a contraction on $\mathcal{H}$. Let $\{c_k\}_{k=-\infty}^{\infty}$ be a sequence of complex numbers with $\overline{\lim}_{|k|\to\infty} |c_k|^{1/|k|} \leq 1$ and assume that the limit*

$$T = \lim_{N\to\infty} \sum_{|k|\leq N} \Big(1 - \frac{|k|}{N+1}\Big) c_k S(k) \quad \textit{in } \mathcal{L}(\mathcal{H})$$

exists in the weak operator topology in $\mathcal{L}(\mathcal{H})$. Then the harmonic function

$$h(z) = \sum_{k=-\infty}^{\infty} c_k r^{|k|} e^{ik\theta}, \quad z = re^{i\theta} \in \mathbb{D}, \tag{5.1}$$

is bounded in absolute value by $\|T\|$.

Proof. The function h is clearly harmonic in $\mathbb{D}$. We have that

$$\begin{aligned}(5.2)\qquad &\langle TK_{\mathcal{H}}(\cdot,z),K_{\mathcal{H}}(\cdot,z)\rangle\\ &= \lim_{N\to\infty}\sum_{|k|\leq N}\Big(1-\frac{|k|}{N+1}\Big)c_k\langle S(k)K_{\mathcal{H}}(\cdot,z),K_{\mathcal{H}}(\cdot,z)\rangle,\quad z\in\mathbb{D}.\end{aligned}$$

A computation using the reproducing property of the kernel function $K_{\mathcal{H}}$ gives that

$$\langle S(k)K_{\mathcal{H}}(\cdot,z),K_{\mathcal{H}}(\cdot,z)\rangle = \langle S^kK_{\mathcal{H}}(\cdot,z),K_{\mathcal{H}}(\cdot,z)\rangle = z^kK_{\mathcal{H}}(z,z),\quad z\in\mathbb{D},$$

for $k\geq 0$, and similarly that

$$\langle S(k)K_{\mathcal{H}}(\cdot,z),K_{\mathcal{H}}(\cdot,z)\rangle = \bar{z}^{|k|}K_{\mathcal{H}}(z,z),\quad z\in\mathbb{D},$$

for $k<0$. We write $z=re^{i\theta}$. By (5.2) we have that

$$\begin{aligned}(5.3)\qquad \langle TK_{\mathcal{H}}(\cdot,z),K_{\mathcal{H}}(\cdot,z)\rangle &= \lim_{N\to\infty}\sum_{|k|\leq N}\Big(1-\frac{|k|}{N+1}\Big)c_kr^{|k|}e^{ik\theta}K_{\mathcal{H}}(z,z)\\ &= h(z)K_{\mathcal{H}}(z,z),\quad z=re^{i\theta}\in\mathbb{D},\end{aligned}$$

since h is harmonic in $\mathbb{D}$. Notice that

$$|\langle TK_{\mathcal{H}}(\cdot,z),K_{\mathcal{H}}(\cdot,z)\rangle| \leq \|T\|\|K_{\mathcal{H}}(\cdot,z)\|^2 = \|T\|K_{\mathcal{H}}(z,z),\quad z\in\mathbb{D},$$

by the Cauchy-Schwarz inequality and the reproducing property of the kernel function $K_{\mathcal{H}}$. By (5.3) we now conclude that $|h(z)|\leq\|T\|$ for all $z\in\mathbb{D}$. This completes the proof of the lemma. □

Remark 5.1. We remark that the assumption $\overline{\lim}_{|k|\to\infty}|c_k|^{1/|k|}\leq 1$ in Lemma 5.1 is redundant in the sense that it follows from the existence of the limit defining the operator $T\in\mathcal{L}(\mathcal{H})$. This follows by Theorem 5.2 below and the uniform boundedness principle.

We can now describe the operators $T\in\mathcal{L}(A_n(\mathbb{D}))$ satisfying equality (0.6) using the functional calculus from Section 4.

Theorem 5.1. *Let $T\in\mathcal{L}(A_n(\mathbb{D}))$ be a bounded linear operator. Then the operator T satisfies equality (0.6) if and only if it has the form of an operator integral*

$$(5.4)\qquad T=\int_{\mathbb{T}}f(e^{i\theta})\,d\omega_{S_n}(e^{i\theta})\quad \textit{in } \mathcal{L}(A_n(\mathbb{D}))$$

of a function $f\in L^\infty(\mathbb{T})$. Furthermore, we have the norm equality $\|T\|=\|f\|_\infty$.

Proof. By Theorem 3.1 and summability considerations from Section 2 we have that an operator $T\in\mathcal{L}(A_n(\mathbb{D}))$ satisfies (0.6) if and only if it has the form of a limit

$$(5.5)\qquad T=\lim_{N\to\infty}\sum_{|k|\leq N}\Big(1-\frac{|k|}{N+1}\Big)c_kS_n(k)\quad\text{in } \mathcal{L}(A_n(\mathbb{D}))$$

with convergence in the strong operator topology in $\mathcal{L}(A_n(\mathbb{D}))$ for some sequence $\{c_k\}_{k=-\infty}^{\infty}$ of complex numbers. By Theorem 4.1 we conclude that every operator T of the form (5.4) satisfies (0.6). Notice also that $\|T\|\leq\|f\|_\infty$ by (1.2) whenever T has the form (5.4).

Let now $T \in \mathcal{L}(A_n(\mathbb{D}))$ be an operator of the form (5.5) with convergence in the strong operator topology. Notice that $|c_k| = \|c_k S_n(k)\| \leq \|T\|$ by homogeneity considerations from Section 2. By Lemma 5.1 the harmonic function h given by (5.1) is such that $|h(z)| \leq \|T\|$ for all $z \in \mathbb{D}$. It is well-known that bounded harmonic functions h in $\mathbb{D}$ correspond to functions f in $L^\infty(\mathbb{T})$ by the Poisson integral formula (see [12, Lemma III.1.2]). Passing to boundary values we obtain a function $f \in L^\infty(\mathbb{T})$ with Fourier coefficients $\widehat{f}(k) = c_k$ for $k \in \mathbb{Z}$ such that $\|f\|_\infty \leq \|T\|$. By Theorem 4.1 this function $f \in L^\infty(\mathbb{T})$ puts the operator T on the form (5.4) with $\|T\| = \|f\|_\infty$. This completes the proof of the theorem. □

Let us return to a Hilbert space $\mathcal{H}$ of analytic functions on the unit disc $\mathbb{D}$ with kernel function $K_\mathcal{H}$. It is easy to see that the linear span of the reproducing elements $K_\mathcal{H}(\cdot, \zeta)$, $\zeta \in \mathbb{D}$, is dense in $\mathcal{H}$, and that the norm of the function

$$f(z) = \sum_{j=1}^{n} c_j K_\mathcal{H}(z, \zeta_j), \quad z \in \mathbb{D}, \tag{5.6}$$

in $\mathcal{H}$, where $\zeta_j \in \mathbb{D}$ for $1 \leq j \leq n$, equals

$$\|f\|^2 = \sum_{j,k=1}^{n} c_j \bar{c}_k K_\mathcal{H}(\zeta_k, \zeta_j) \tag{5.7}$$

(see [5, Section I.2]).

Proposition 5.1. *Let $\mathcal{H}$ be a Hilbert space of analytic functions on $\mathbb{D}$ such that the shift operator $S : f \mapsto zf$ acts as a contraction on $\mathcal{H}$. Then $\lim_{k\to\infty} S^{*k} = 0$ in the strong operator topology in $\mathcal{L}(\mathcal{H})$. In particular, the operator measure $d\omega_S$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{T}$.*

Proof. A computation shows that the adjoint shift operator S^* acts as

$$S^* K_\mathcal{H}(\cdot, \zeta) = \bar{\zeta} K_\mathcal{H}(\cdot, \zeta), \quad \zeta \in \mathbb{D},$$

on reproducing elements. Let $f \in \mathcal{H}$ be a function of the form (5.6), and notice that

$$\|S^{*m} f\|^2 = \sum_{j,k=1}^{n} c_j \bar{\zeta}_j^m \bar{c}_k \zeta_k^m K_\mathcal{H}(\zeta_k, \zeta_j) \to 0 \quad \text{as } m \to \infty$$

by (5.7). Since $\|S^{*m}\| \leq 1$ for $m \geq 0$, we conclude by an approximation argument that $\lim_{m\to\infty} S^{*m} = 0$ in the strong operator topology in $\mathcal{L}(\mathcal{H})$. The absolute continuity of $d\omega_S$ is now evident by earlier remarks. □

We shall next compute the norm of the operator $\int_\mathbb{T} f d\omega_S$.

Theorem 5.2. *Let $\mathcal{H}$ be a Hilbert space of analytic functions on $\mathbb{D}$ such that the shift operator $S : f \mapsto zf$ acts as a contraction on $\mathcal{H}$. Then a measurable function f on $\mathbb{T}$ is integrable with respect to $d\omega_S$ if and only if $f \in L^\infty(\mathbb{T})$. Furthermore, we have the norm equality*

$$\| \int_\mathbb{T} f(e^{i\theta}) d\omega_S(e^{i\theta}) \| = \|f\|_\infty, \quad f \in L^\infty(\mathbb{T}). \tag{5.8}$$

Proof. Let $f \in L^\infty(\mathbb{T})$. Recall that by (1.2) we always have that $\| \int_{\mathbb{T}} f d\omega_S \| \leq \|f\|_\infty$. By Theorem 4.1 we know that

$$\int_{\mathbb{T}} f(e^{i\theta}) d\omega_S(e^{i\theta}) = \lim_{N\to\infty} \sum_{|k|\leq N} \Big(1 - \frac{|k|}{N+1}\Big) \widehat{f}(k) S(k) \quad \text{in } \mathcal{L}(\mathcal{H})$$

with convergence in the strong operator topology in $\mathcal{L}(\mathcal{H})$, where $\widehat{f}(k)$ is the k-th Fourier coefficient of $f \in L^\infty(\mathbb{T})$. By Lemma 5.1 we conclude that the Poisson integral $h = P[f]$ of f is bounded in absolute value by $\| \int_T f d\omega_T \|$. This proves the norm equality (5.8). The integrability assertion also follows by equality (5.8). This completes the proof of the theorem. □

We mention that the norm equality (5.8) in Theorem 5.2 also follows by a result of Conway and Ptak [9, Theorem 2.2] using that $d\omega_S$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{T}$ and that the $H^\infty(\mathbb{D})$-functional calculus for the shift operator S is isometric.

Proposition 5.2. *Let $\mathcal{H}$ be a Hilbert space of analytic functions on $\mathbb{D}$ such that the shift operator $S : f \mapsto zf$ acts as a contraction on $\mathcal{H}$ and let $f \in L^\infty(\mathbb{T})$. Then*

$$\langle \int_{\mathbb{T}} f(e^{i\theta}) d\omega_S(e^{i\theta}) K_{\mathcal{H}}(\cdot, z), K_{\mathcal{H}}(\cdot, z)\rangle = P[f](z) K_{\mathcal{H}}(z,z), \quad z \in \mathbb{D},$$

where $P[f]$ is the Poisson integral of f.

Proof. By Theorem 4.1 we have that

$$\int_{\mathbb{T}} f(e^{i\theta}) d\omega(e^{i\theta}) = \lim_{N\to\infty} \sum_{|k|\leq N} \Big(1 - \frac{|k|}{N+1}\Big) \widehat{f}(k) S(k) \quad \text{in } \mathcal{L}(\mathcal{H})$$

with convergence in the strong operator topology in $\mathcal{L}(\mathcal{H})$. A computation similar to the one in Lemma 5.1 now gives that

$$\begin{aligned}
&\langle \int_{\mathbb{T}} f(e^{i\theta}) d\omega(e^{i\theta}) K_{\mathcal{H}}(\cdot, z), K_{\mathcal{H}}(\cdot, z)\rangle \\
&\qquad = \lim_{N\to\infty} \sum_{|k|\leq N} \Big(1 - \frac{|k|}{N+1}\Big) \widehat{f}(k) \langle S(k) K_{\mathcal{H}}(\cdot, z), K_{\mathcal{H}}(\cdot, z)\rangle \\
&\qquad = P[f](z) K_{\mathcal{H}}(z,z), \quad z \in \mathbb{D}.
\end{aligned}$$

This completes the proof of the proposition. □

We remark that the above results Theorem 5.2 and Proposition 5.2 give converses to the summability results for the functional calculus in Section 4: If the limit

$$T = \lim_{N\to\infty} \sum_{|k|\leq N} \Big(1 - \frac{|k|}{N+1}\Big) c_k S(k) \quad \text{in } \mathcal{L}(\mathcal{H}) \tag{5.9}$$

exists in the weak operator topology in $\mathcal{L}(\mathcal{H})$, then $T = \int_{\mathbb{T}} f d\omega_S$ in $\mathcal{L}(\mathcal{H})$ for the function $f \in L^\infty(\mathbb{T})$ with Fourier coefficients $\widehat{f}(k) = c_k$ for $k \in \mathbb{Z}$. Similarly, if the limit (5.9) exists in the uniform operator topology in $\mathcal{L}(\mathcal{H})$, then $T = \int_{\mathbb{T}} f d\omega_S$ in $\mathcal{L}(\mathcal{H})$ with $f \in C(\mathbb{T})$. We omit the details.

We shall next discuss compactness of an operator of the form $\int_{\mathbb{T}} f d\omega_S$. We shall need the following off-diagonal estimate for a reproducing kernel function.

Lemma 5.2. *Let $\mathcal{H}$ be a Hilbert space of analytic functions on $\mathbb{D}$ such that the shift operator $S : f \mapsto zf$ acts as a contraction on $\mathcal{H}$ and denote by $K_{\mathcal{H}}$ the reproducing kernel function for $\mathcal{H}$. Then*

$$|K_{\mathcal{H}}(z,\zeta)|^2 \leq \frac{(1-|z|^2)(1-|\zeta|^2)}{|1-\bar{\zeta}z|^2} K_{\mathcal{H}}(z,z)K_{\mathcal{H}}(\zeta,\zeta), \quad (z,\zeta) \in \mathbb{D}\times\mathbb{D}.$$

Proof. The essential property needed for a function to be the kernel function for a reproducing kernel Hilbert space is that of positive definiteness (see [5, Section I.2]). The assumption that the shift operator S is a contraction on $\mathcal{H}$ can equivalently be formulated saying that the function

$$L(z,\zeta) = (1-\bar{\zeta}z)K_{\mathcal{H}}(z,\zeta), \quad (z,\zeta) \in \mathbb{D}\times\mathbb{D},$$

is positive definite on $\mathbb{D}\times\mathbb{D}$ (see [5, Section I.7]). By the Cauchy-Schwarz inequality we have that

$$|L(z,\zeta)| \leq L(z,z)^{1/2}L(\zeta,\zeta)^{1/2}, \quad (z,\zeta) \in \mathbb{D}\times\mathbb{D}.$$

This last inequality gives the conclusion of the lemma. □

We remark that there is a more refined off-diagonal estimate for kernel functions related to the operator inequality

$$\|Sf+g\|^2 \leq 2(\|f\|^2 + \|Sg\|^2), \quad f,g \in \mathcal{H},$$

for the shift operator S on $\mathcal{H}$ (see [18, Proposition 4.5] and [10, Section 6]).

Corollary 5.1. *Let $\mathcal{H}$ be a Hilbert space of analytic functions on $\mathbb{D}$ such that the shift operator $S : f \mapsto zf$ acts as a contraction on $\mathcal{H}$. Then the normalized reproducing elements*

$$K_\zeta = K_{\mathcal{H}}(\cdot,\zeta)/K_{\mathcal{H}}(\zeta,\zeta)^{1/2}, \quad \zeta \in \mathbb{D}, \quad K_{\mathcal{H}}(\zeta,\zeta) \neq 0,$$

converge weakly to zero in $\mathcal{H}$ as $|\zeta| \to 1$.

Proof. Notice that $\langle K_\zeta, K_{\mathcal{H}}(\cdot,z)\rangle = K_{\mathcal{H}}(z,\zeta)/K_{\mathcal{H}}(\zeta,\zeta)^{1/2}$, and recall the standard formula

$$1 - \left|\frac{z-\zeta}{1-\bar{\zeta}z}\right|^2 = \frac{(1-|z|^2)(1-|\zeta|^2)}{|1-\bar{\zeta}z|^2}, \quad (z,\zeta) \in \mathbb{D}\times\mathbb{D},$$

which can be proved by straightforward computation. By the estimate in Lemma 5.2 we conclude that $\langle K_\zeta, K_{\mathcal{H}}(\cdot,z)\rangle \to 0$ as $|\zeta| \to 1$ for every $z \in \mathbb{D}$. Taking linear combinations we see that $\langle K_\zeta, f\rangle \to 0$ as $|\zeta| \to 1$ for every function $f \in \mathcal{H}$ of the form (5.6). Since $\|K_\zeta\| = 1$ for every $\zeta \in \mathbb{D}$ by construction, the conclusion of the corollary follows by a standard approximation argument. □

We can now show that the operator $\int_{\mathbb{T}} f d\omega_S$ is compact only when $f = 0$.

Theorem 5.3. *Let $\mathcal{H}$ be a Hilbert space of analytic functions on $\mathbb{D}$ such that the shift operator $S : f \mapsto zf$ acts as a contraction on $\mathcal{H}$, and let $f \in L^\infty(\mathbb{T})$. Then the operator*

$$\int_{\mathbb{T}} f(e^{i\theta})d\omega_S(e^{i\theta}) \quad \text{in } \mathcal{L}(\mathcal{H})$$

is compact if and only if $f = 0$.

Proof. It is evident that the zero operator is compact. We turn to the reverse implication. By Proposition 5.2 we have that

$$\langle \int_{\mathbb{T}} f(e^{i\theta}) d\omega_S(e^{i\theta}) K_z, K_z\rangle = P[f](z), \quad z \in \mathbb{D}, \quad K_{\mathcal{H}}(z,z) \neq 0,$$

where $K_z = K_{\mathcal{H}}(\cdot, z)/K_{\mathcal{H}}(z,z)^{1/2}$ for $z \in \mathbb{D}$, $K_{\mathcal{H}}(z,z) \neq 0$, are the normalized reproducing elements for $\mathcal{H}$ and $P[f]$ is the Poisson integral of f. By Corollary 5.1 we know that $K_z \to 0$ weakly in $\mathcal{H}$ as $|z| \to 1$. By compactness of the operator $\int_{\mathbb{T}} f d\omega_S$ we conclude that $\lim_{|z|\to 1} P[f](z) = 0$. This proves that $f = 0$. □

6. Toeplitz operators. Harmonic symbols

From the point of view of function theory a Toeplitz operator on the Bergman space $A_n(\mathbb{D})$, $n \geq 2$, is an integral operator T_h acting as

$$(T_h f)(z) = \int_{\bar{\mathbb{D}}} \frac{1}{(1-\bar{\zeta}z)^n} h(\zeta) f(\zeta) d\mu_n(\zeta), \quad z \in \mathbb{D}, \tag{6.1}$$

on, say, polynomials $f \in A_n(\mathbb{D})$ defined using a function h in $L^1(\mathbb{D}, d\mu_n)$ so that the above integral (6.1) makes sense. The function h is called the symbol for the operator T_h, and the function

$$K_n(z,\zeta) = \frac{1}{(1-\bar{\zeta}z)^n}, \quad (z,\zeta) \in \mathbb{D} \times \mathbb{D},$$

appearing in (6.1) is the reproducing kernel function for the space $A_n(\mathbb{D})$. For symbols h that are harmonic in $\mathbb{D}$ the properties of boundedness and compactness of the operator T_h defined by (6.1) are easily settled: The operator T_h is bounded as an operator on $A_n(\mathbb{D})$ if and only if the symbol h is bounded and $\|T_h\| = \sup_{z\in\mathbb{D}} |h(z)|$. The operator T_h is compact in $\mathcal{L}(A_n(\mathbb{D}))$ if and only if $h(z) = 0$ for all $z \in \mathbb{D}$ (see [20, Section 6.1]).

In the case of the Hardy space $A_1(\mathbb{D})$ we can identify a bounded harmonic function h with its boundary value function in $L^\infty(\mathbb{T})$ (see [12, Lemma III.1.2]) to make (6.1) a standard formula for the action of a Hardy space Toeplitz operator.

The matrix representation of a Toeplitz operator T_h relative to the orthogonal basis of monomials is easily computed.

Lemma 6.1. *Let $e_k(z) = z^k$ for $k \geq 0$, and let h be a bounded harmonic function in $\mathbb{D}$ with power series expansion*

$$h(z) = \sum_{k=-\infty}^{\infty} c_k r^{|k|} e^{ik\theta}, \quad z = re^{i\theta} \in \mathbb{D}. \tag{6.2}$$

Then

$$\langle T_h e_k, e_j\rangle_{A_n} = c_{j-k}\mu_{n;j}$$

for $j,k \geq 0$. The matrix representation of the Toeplitz operator T_h relative to the orthogonal basis of monomials $\{e_k\}_{k\geq 0}$ thus has the form $\{c_{j-k}\}_{j,k\geq 0}$.

Proof. Notice that the operator T_h is obtained by a multiplication by h followed by an orthogonal projection onto $A_n(\mathbb{D})$. We have that

$$\langle T_h e_k, e_j\rangle_{A_n} = \int_{\bar{\mathbb{D}}} h(z) z^k \bar{z}^j d\mu_n(z) = c_{j-k} \int_{\bar{\mathbb{D}}} |z|^{2j} d\mu_n(z) = c_{j-k}\mu_{n;j}$$

for $j,k \geq 0$. This completes the proof of the lemma. □

We can now characterize Toeplitz operators T_h with harmonic symbols h using equality (0.6).

Theorem 6.1. *Let $T \in \mathcal{L}(\mathcal{H})$ be a bounded linear operator. Then T satisfies equality (0.6) if and only if $T = T_h$ is a Toeplitz operator on $A_n(\mathbb{D})$ with bounded harmonic symbol h.*

Proof. We first show that the m-th homogeneous part $(T_h)_m$ of the Toeplitz operator T_h with bounded harmonic symbol h given by (6.2) is equal to $c_m S_n(m)$ for $m \in \mathbb{Z}$. By Theorem 3.1 this will then show that every such Toeplitz operator T_h satisfies equality (0.6).

Let us proceed to details. We consider first the case $m \geq 0$. By Lemma 6.1 and considerations from Section 2 we have that m-th homogeneous part $(T_h)_m$ of T_h is given by the matrix $\{\delta_{m,j-k} c_{j-k}\}_{j,k\geq 0}$ relative to the orthogonal basis $\{e_k\}_{k\geq 0}$ of monomials. This means that the operator $(T_h)_m$ acts as

$$(T_h)_m e_k = c_m e_{m+k} = c_m S_n^m e_k$$

for $k \geq 0$. We conclude that $(T_h)_m = c_m S_n^m$ for $m \geq 0$. We consider next the case $m < 0$. Notice that $(T_h)_m^*$ is the $-m$-th homogeneous part of $T_h^* = T_{\bar{h}}$. Since $-m > 0$, we have by the case already treated that $(T_h)_m^* = \bar{c}_m S_n^{|m|}$. Taking adjoints we see that $(T_h)_m = c_m S_n^{*|m|}$ for $m < 0$.

Let now $T \in \mathcal{L}(A_n(\mathbb{D}))$ be an operator satisfying (0.6). By Theorem 3.1 the m-th homogeneous part T_m of T has the form $T_m = c_m S_n(m)$ for $m \in \mathbb{Z}$, where c_m is a complex number. Notice also that $|c_m| = \|T_m\| \leq \|T\|$ for $m \in \mathbb{Z}$. By considerations from Section 2 the operator T can be reconstructed from its homogeneous parts $T_m = c_m S_n(m)$ in the sense that the limit

$$T = \lim_{N\to\infty} \sum_{|m|\leq N} \Big(1 - \frac{|m|}{N+1}\Big) c_m S_n(m) \quad \text{in } \mathcal{L}(A_n(\mathbb{D}))$$

exists in the strong operator topology in $\mathcal{L}(A_n(\mathbb{D}))$. By Lemma 5.1 the harmonic function h given by (6.2) is bounded in absolute value by $\|T\|$. We can now consider the Toeplitz operator T_h in $\mathcal{L}(A_n(\mathbb{D}))$ with this harmonic function h as symbol. By the first part of the proof this Toeplitz operator T_h has the same m-th homogeneous part $c_m S_n(m)$ as the operator T for every $m \in \mathbb{Z}$. We conclude that these two operators must be equal: $T = T_h$. This completes the proof of the theorem. □

7. General Toeplitz operators

As we have indicated in the introduction balayage considerations seem to suggest that Bergman space Toeplitz operators should be considered as integrals using naturally associated positive operator measures. We shall indicate in this section some rudiments of such a theory.

Toeplitz operators arise naturally when the functional calculus for normal operators is compressed down to a subspace of the original space and the operators obtained in this way have in many cases properties such as boundedness or compactness for more general classes of symbols compared to the original functional calculus for normal operators. The functional calculus for normal operators is defined by integration with respect to the associated spectral measure and we are led to consider integrability properties of compressed spectral measures.

Let $\mathcal{K}$ be a Hilbert space and let dE be an $\mathcal{L}(\mathcal{K})$-valued spectral measure on a measure space $(X, \mathfrak{S})$. Let $\mathcal{H}$ be a closed subspace of $\mathcal{K}$ and consider the $\mathcal{L}(\mathcal{H})$-valued set function μ defined by

$$\mu(S) = P_{\mathcal{H}} E(S)|_{\mathcal{H}}, \quad S \in \mathfrak{S}, \tag{7.1}$$

where $P_{\mathcal{H}}$ is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$. It is apparent that $d\mu$ is a positive operator measure such that $\mu(X) = I$. Conversely, if we are given a positive operator measure $d\mu$ on X such that $\mu(X) = I$, then a construction of Neumark [14] generalizing that of an L^2-space produces a larger Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ as a closed subspace and an $\mathcal{L}(\mathcal{K})$-valued spectral measure dE such that (7.1) holds. This result of Neumark [14] is often referred to as the Neumark dilation theorem.

A special case often encountered in analysis is when dE is the canonical spectral measure associated to an L^2-space. Let $d\nu$ be a positive (scalar) measure on X, and consider the associated L^2-space $L^2(d\nu)$. The prototype of a spectral measure dE is given by

$$E(S)\varphi = \chi_S \varphi, \quad \varphi \in L^2(d\nu), \quad S \in \mathfrak{S},$$

where χ_S is the characteristic function for the set S. Let now $\mathcal{H}$ be a closed subspace of $L^2(d\nu)$, and consider the $\mathcal{L}(\mathcal{H})$-valued operator measure $d\mu_{\mathcal{H}} = d\mu$ defined by (7.1). The marginal distributions $d(\mu_{\mathcal{H}})_{\varphi,\psi}$ of $d\mu_{\mathcal{H}}$ are easily computed as

$$d(\mu_{\mathcal{H}})_{\varphi,\psi} = \varphi\bar{\psi} d\nu, \quad \varphi, \psi \in \mathcal{H}.$$

We record the following proposition.

Proposition 7.1. *Let $\mathcal{H}$ be a closed subspace of $L^2(d\nu)$, let f be a measurable function on X and set $d\lambda = |f| d\nu$. Then f is integrable with respect to $d\mu_{\mathcal{H}}$ if and only if the embedding*

$$\int_X |\varphi|^2 d\lambda \leq C \int_X |\varphi|^2 d\nu, \quad \varphi \in \mathcal{H}, \tag{7.2}$$

of $\mathcal{H}$ into $L^2(d\lambda)$ is continuous. Furthermore, the following statements hold true:

1. *We have the norm bound $\| \int_X f d\mu_{\mathcal{H}} \| \leq C$ with equality if f is non-negative, where C is the best constant in (7.2).*
2. *If the embedding (7.2) is compact, then $\int_X f d\mu_{\mathcal{H}}$ is a compact operator in $\mathcal{L}(\mathcal{H})$.*
3. *If f is non-negative and $\int_X f d\mu_{\mathcal{H}}$ is a compact operator in $\mathcal{L}(\mathcal{H})$, then the embedding (7.2) is compact.*

Sketch of proof. If f is integrable with respect to $d\mu_{\mathcal{H}}$, then $\int_X |\varphi|^2 d\lambda < \infty$ for every $\varphi \in \mathcal{H}$, showing that (7.2) holds by the Banach-Steinhaus theorem. Conversely, if (7.2) holds we have by the Cauchy-Schwarz inequality that

$$\int_X |f||\varphi||\psi| d\nu \leq \Big(\int_X |\varphi|^2 |f| d\nu \Big)^{1/2} \Big(\int_X |\psi|^2 |f| d\nu \Big)^{1/2} \leq C \|\varphi\| \|\psi\|, \quad \varphi, \psi \in \mathcal{H},$$

showing that f is integrable with respect to $d\mu_{\mathcal{H}}$, and that $\| \int_X f d\mu_{\mathcal{H}} \| \leq C$. We notice also that this last estimation is the best possible for $\int_X f \varphi \bar{\psi} d\mu$ if f is non-negative.

We consider next the compactness properties of $T_f = \int_X f d\mu_{\mathcal{H}}$. Denote by R the embedding (7.2) of $\mathcal{H}$ into $L^2(d\lambda)$. Let $h = \operatorname{sgn}(f)$, where $\operatorname{sgn}(z) = z/|z|$ for $z \neq 0$

and $\operatorname{sgn}(0) = 0$, and denote by H the operator on $L^2(d\lambda)$ given by multiplication by h. Notice that the operator $T_f = \int_X f d\mu_{\mathcal{H}}$ factorizes as

$$T_f = R^*HR \quad \text{in } \mathcal{L}(\mathcal{H}).$$

This last equality shows that T_f is compact if R is compact. Notice that $T_f = R^*R$ if f is non-negative. If T_f is also compact and $\varphi_j \to 0$ weakly in $\mathcal{H}$, then $\langle T_f\varphi_j, \varphi_j\rangle = \|R\varphi_j\|^2 \to 0$ showing that R is compact. □

The above Proposition 7.1 gives easy criteria for existence as an operator integral, norm bounds and compactness of a generalized Toeplitz operator T_f defined as an $\mathcal{L}(\mathcal{H})$-valued operator integral by $T_f = \int_X f d\mu_{\mathcal{H}}$. If the space $\mathcal{H}$ has bounded point evaluations

$$\mathcal{H} \ni f \mapsto f(x), \quad x \in \Omega,$$

on a set Ω standard duality arguments gives the existence of reproducing elements $K_{\mathcal{H}}(\cdot, x) \in \mathcal{H}$ for $x \in \Omega$ such that

$$f(x) = \langle \varphi, K_{\mathcal{H}}(\cdot, x)\rangle, \quad x \in \Omega,$$

for $\varphi \in \mathcal{H}$. In terms of these reproducing elements the action of the Toeplitz operator $T_f = \int_X f d\mu_{\mathcal{H}}$ is given by

$$(T_f\varphi)(x) = \langle T_f\varphi, K_{\mathcal{H}}(\cdot, x)\rangle = \int_X \overline{K_{\mathcal{H}}(\cdot, x)} f\varphi d\nu, \quad x \in \Omega,$$

for $\varphi \in \mathcal{H}$.

In complex analysis embeddings of the form (7.2) appear under the name of Carleson embeddings and in the context of standard weighted Bergman spaces boundedness and compactness of such embeddings have well-known descriptions using hyperbolic discs or so-called Carleson squares (see [20, Section 6.2]). In the context of the Bergman spaces $A_n(\mathbb{D})$ the natural regularity class for symbols f of Toeplitz operators T_f on $A_n(\mathbb{D})$ suggested here is measurable functions f on $\mathbb{D}$ such that $|f|d\mu_n$ is a Bergman space Carleson measure for $A_n(\mathbb{D})$. We mention that this regularity class for symbols appears in a recent result of Miao and Zheng [13] characterizing the compact Toeplitz operators on the unweighted Bergman space $A_2(\mathbb{D})$ in terms of vanishing of the so-called Berezin transform.

A common property possessed by all Bergman shifts is that of being a subnormal operator. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be subnormal if it is part of a normal operator $N \in \mathcal{L}(\mathcal{K})$ (restriction to an invariant subspace); the operator $N \in \mathcal{L}(\mathcal{K})$ is then called a normal extension of T. Associated to a subnormal operator $T \in \mathcal{L}(\mathcal{H})$ we have the positive $\mathcal{L}(\mathcal{H})$-valued operator measure $d\mu_T = d\mu$ defined by (7.1), where dE is the spectral measure for a normal extension $N \in \mathcal{L}(\mathcal{K})$ of T. This operator measure $d\mu_T$ is uniquely characterized by the property that

$$T^{*j}T^k = \int_X \bar{z}^j z^k d\mu_T(z) \quad \text{in } \mathcal{L}(\mathcal{H}) \tag{7.3}$$

for $j, k \geq 0$ (see [8, Section II.1]).

We recall that the balayage of a complex measure $d\mu$ on $\bar{\mathbb{D}}$ onto $\mathbb{T} = \partial\mathbb{D}$ is the complex measure $d\mu'$ on $\mathbb{T}$ defined by

$$\int_{\mathbb{T}} \varphi(e^{i\theta}) d\mu'(e^{i\theta}) = \int_{\bar{\mathbb{D}}} P[\varphi](z) d\mu(z), \quad \varphi \in C(\mathbb{T}),$$

where $P[\varphi]$ is the Poisson integral of φ. A straightforward argument using Fubini's theorem shows that the balayage of $d\mu$ is given by the $L^1(\mathbb{T})$-function

$$P^*[d\mu](e^{i\theta}) = \int_{\mathbb{D}} P(z, e^{i\theta}) d\mu(z), \quad e^{i\theta} \in \mathbb{T}, \tag{7.4}$$

if the mass of $d\mu$ is carried by the open unit disc $\mathbb{D}$.

We make the following observation.

Proposition 7.2. *Let $T \in \mathcal{L}(\mathcal{H})$ be a subnormal contraction and let $d\mu_T = d\mu$ be the associated positive operator measure in $\bar{\mathbb{D}}$ given by (7.1) or (7.3). Then*

$$\int_{\bar{\mathbb{D}}} u(z) d\mu_T(z) = \int_{\mathbb{T}} u(e^{i\theta}) d\omega_T(e^{i\theta}) \tag{7.5}$$

for every function $u \in C(\bar{\mathbb{D}})$ which is harmonic in $\mathbb{D}$ meaning that $d\omega_T$ is the balayage of $d\mu_T$ onto $\mathbb{T} = \partial\mathbb{D}$. In particular, if the mass of $d\mu_T$ is concentrated in the open unit disc $\mathbb{D}$, then $d\omega_T$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{T}$.

Proof. That the operator measures $d\mu_T$ and $d\omega_T$ have the same action on harmonic polynomials is evident by (7.3) and the description of $d\omega_T$ in Section 4. Equality (7.5) follows by an approximation argument. By (7.5) the complex measure $d(\omega_T)_{x,y}$ is the balayage of $d(\mu_T)_{x,y}$ onto $\mathbb{T} = \partial\mathbb{D}$ giving the last assertion of the proposition. □

We mention that for a more general positive operator measure $d\mu$ carried by the open unit disc $\mathbb{D}$ the operator integral in (7.4) no longer exists in our sense as an operator in $\mathcal{L}(\mathcal{H})$.

We wish to mention here also that Toeplitz operators defined as integrals of positive operator measures has been studied by Aleman [3, Section III.4] in the context of subnormal shift operators in Hilbert spaces of analytic functions; see also Aleman [4].

References

[1] P. Ahern, On the range of the Berezin transform, *J. Funct. Anal.* **215** (2004) 206-216.

[2] P. Ahern and Ž. Čučković, A theorem of Brown-Halmos type for Bergman space Toeplitz operators, *J. Funct. Anal.* **187** (2001) 200-210.

[3] A. Aleman, *The multiplication operators on Hilbert spaces of analytic functions*, Habilitationsschrift, Fernuniversität Hagen, 1993.

[4] A. Aleman, Subnormal operators with compact selfcommutator, *Manuscripta Math.* **91** (1996) 353-367.

[5] N. Aronszajn, Theory of reproducing kernels, *Trans. Amer. Math. Soc.* **68** (1950) 337-404.

[6] S. Axler and Ž. Čučković, Commuting Toeplitz operators with harmonic symbols, *Integral Equations Operator Theory* **14** (1991) 1-12.

[7] A. Brown and P.R. Halmos, Algebraic properties of Toeplitz operators, *J. Reine Angew. Math.* **213** (1963/1964) 89-102.

[8] J.B. Conway, *The theory of subnormal operators*, American Mathematical Society, 1991.

[9] J.B. Conway and M. Ptak, The harmonic functional calculus and hyperreflexivity, *Pacific J. Math.* **204** (2002) 19-29.

[10] H. Hedenmalm, S. Jakobsson, and S.M. Shimorin, A biharmonic maximum principle for hyperbolic surfaces, *J. Reine Angew. Math.* **550** (2002) 25-75.

[11] H. Hedenmalm, B. Korenblum and K. Zhu, *Theory of Bergman spaces*, Springer, 2000.

[12] Y. Katznelson, *An introduction to harmonic analysis*, Dover Publications, 1976.

[13] J. Miao and D. Zheng, Compact operators on Bergman spaces, *Integral Equations Operator Theory* **48** (2004) 61-79.

[14] M.A. Neumark, On a representation of additive operator set functions, *C. R. (Doklady) Acad. Sci. URSS (N.S.)* **41** (1943) 359-361.
[15] A. Olofsson, Operator-valued n-harmonic measure in the polydisc, *Studia Math.* **163** (2004) 203-216.
[16] A. Olofsson, An operator-valued Berezin transform and the class of n-hypercontractions, submitted for publication.
[17] A. Olofsson, A characteristic operator function for the class of n-hypercontractions, *J. Funct. Anal.* **236** (2006) 517-545.
[18] S.M. Shimorin, Wold-type decompositions and wandering subspaces of operators close to isometries, *J. Reine Angew. Math.* **531** (2001) 147-189.
[19] B. Sz.-Nagy and C. Foias, *Harmonic analysis of operators on Hilbert space*, North-Holland, 1970.
[20] K. Zhu, *Operator theory in function spaces*, Marcel Dekker, 1990.

Issam Louhichi, Department of Mathematics, Mail Stop 942, The University of Toledo, 2801 W. Bancroft St., Toledo OH 43606-3390, USA
E-mail address: issam.louhichi2@utoledo.edu

Anders Olofsson, Falugatan 22 1tr, SE-113 32 Stockholm, Sweden
E-mail address: ao@math.kth.se