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POWERS AND ROOTS OF TOEPLITZ OPERATORS

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ABSTRACT. We study the commutativity of two Toeplitz operators whose symbols are quasihomogeneous functions. We give a relationship between this commutativity and the roots (or powers) of the Toeplitz operators (Proposition 7). We use this to characterize Toeplitz operators with symbols in $L^\infty(\mathbb{D})$ which commute with Toeplitz operators whose symbols are of the form $e^{ip\theta}r^m$ (Theorem 13).

1. Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$, and let dA denote normalized Lebesgue area measure. The Bergman space, denoted by L^2_a, is the Hilbert space of analytic functions on $\mathbb{D}$ that are square integrable with respect to dA. It is well known that L^2_a is a closed subspace of the Hilbert space $L^2(\mathbb{D}, dA)$ and $(\sqrt{n+1}z^n)_{n\in\mathbb{N}}$ is an orthonormal basis of L^2_a. Let P be the orthogonal projection of $L^2(\mathbb{D}, dA)$ onto L^2_a. For a function $\phi \in L^\infty(\mathbb{D}, dA)$, the Toeplitz operator with symbol ϕ is the operator T_ϕ from L^2_a to L^2_a defined by $T_\phi(f) = P(\phi f)$.

If $k_z(w) = \frac{1}{(1-\bar{z}w)^2} = \sum_{j=0}^{\infty}(1+j)w^j\bar{z}^j$ is the Bergman reproducing kernel, then

$$T_\phi(f)(z) = P(\phi f)(z) = \int_{\mathbb{D}} \phi(w)f(w)\overline{k_z(w)}\, dA(w).$$

The question to be studied in this paper is : When do two Toeplitz operators T_ϕ and T_ψ commute? In 1964, Brown and Halmos [4] solved this problem for the analogously defined Toeplitz operators on the Hardy space. They showed that $T_\phi T_\psi = T_\psi T_\phi$ for some ϕ and $\psi \in L^\infty(\mathbb{T})$, where $\mathbb{T}$ is the unit circle of $\mathbb{C}$, if and only if either

(a) ϕ and ψ are both analytic,
or
(b) $\bar{\phi}$ and $\bar{\psi}$ are both analytic,
or
(c) one of the two symbols is a linear function of the other.

We recall that a function in $L^\infty(\mathbb{T})$ is said to be analytic if all of its Fourier coefficients with negative indices are equal to 0.

The same question concerning Toeplitz operators on the Bergman space has a much more complicated answer. There are however some results which resemble those of [4]. In fact, Axler and Čučković proved in [2] that the condition that one of (a), (b) or (c) be true is still necessary and sufficient when the two symbols ϕ

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and ψ are bounded harmonic functions on $\mathbb{D}$. Moreover, with Rao [3], they proved that if ϕ is a bounded analytic function and if ψ is a bounded symbol such that T_ϕ and T_ψ commute then ψ must be analytic too. When we consider arbitrary symbols, things are different. In [5] Čučković and Rao used the Mellin transform to study the commutativity of multiplication of two Toeplitz operators T_ϕ and T_ψ on the Bergman space and describe those operators which commute with $T_{e^{ip\theta}r^m}$ for $(m,p) \in \mathbb{N}\times\mathbb{N}$. In this paper we use our results from [7] to interpret and extend the results of [5]. We give some solutions in the case where the Toeplitz operators have symbols which are "quasihomogeneous" functions and show that these solutions are related to "p^{th} roots" and powers of the Toeplitz operators.
As in [7] we say that a bounded symbol f is quasihomogeneous of degree k if it is of the form $e^{ik\theta}\phi$ where ϕ is a radial function. In this case we say that the Toeplitz operator T_f is quasihomogeneous of degree k.

2. Preliminaries

The Mellin transform of a function $\psi \in L^1([0,1], rdr)$ is defined by

$$\widehat{\psi}(z) = \int_0^1 \psi(r) r^{z-1}\, dr$$

It is easy to see that $\widehat{\psi}$ is a bounded holomorphic function on the half-plane $\Pi = \{z : \Re z > 2\}$.
We denote the Mellin convolution of two functions ϕ and ψ by $\phi *_M \psi$ and we define it by the equation :

$$(\phi *_M \psi)(r) = \int_r^1 \phi(\frac{r}{t})\psi(t)\frac{dt}{t}.$$

It is clear that the Mellin transform converts Mellin convolution into a pointwise product, i.e that :

$$(1) \qquad \widehat{(\phi *_M \psi)}(r) = \widehat{\phi}(r)\widehat{\psi}(r)$$

We shall often use the following classical theorem (see [8, p. 102]).

Theorem 1. *Suppose that f is a bounded, holomorphic function on $\{z : \Re z > 0\}$ which vanishes at the pairwise distinct points $d_1, d_2 \cdots$, where*

i) $\inf\{|d_n|\} > 0$
and
ii) $\sum_{n\geq 1} \Re(\frac{1}{d_n}) = \infty$.

Then f vanishes identically on $\{z : \Re z > 0\}$.

Remark 2. *We shall often apply this theorem to show that :
if $\psi \in L^1([0,1], rdr)$ and if there exist $n_0 \in \mathbb{Z}_+, p \in \mathbb{N}$ such that*

$$\widehat{\psi}(n_0 + pk) = 0 \textit{ for all } k \in \mathbb{N},$$

then $\widehat{\psi}(z) = 0$ for all $z \in \{z : \Re z > 2\}$ and so $\psi = 0$.

3. Powers of Toeplitz operators

The following Lemma determines the values of powers of a bounded quasihomogeneous Toeplitz operator evaluated at any element of the orthonormal basis of L^2_a.

Lemma 3. *Let $n \in \mathbb{N}, s \in \mathbb{Z}_+$ and let ψ be a bounded radial function on $\mathbb{D}$. Then, for all $k \in \mathbb{N}$ we have*

$$\begin{aligned}\Big(T_{e^{is\theta}\psi}\Big)^n(\xi^k)(z) &= \Big[\prod_{j=0}^{n-1} 2(k+js+s+1)\widehat{\psi}\big(2k+2js+s+2\big)\Big]z^{k+ns} \\ &= \frac{\prod_{j=0}^{n-1}\widehat{\psi}\big(2k+2js+s+2\big)}{\prod_{j=0}^{n-1}\widehat{\mathbb{1}}(2k+2js+2s+2)}z^{k+ns},\end{aligned}$$

where $\mathbb{1}$ denotes the constant function with value one.

PROOF. The lemma is a consequence of the following direct calculation : we write

$$T_{e^{is\theta}\psi}(\xi^k)(z) = \int_0^1\int_0^{2\pi}\psi(r)r^k\sum_{j=0}^{\infty}(j+1)e^{i(k+s-j)\theta}r^jz^j\frac{1}{\pi}\,rdrd\theta$$

and interchange the integral over $[0, 2\pi]$ and the sum to see that

$$\begin{aligned}T_{e^{is\theta}\psi}(\xi^k)(z) &= 2(k+s+1)\widehat{\psi}(2k+s+2)z^{k+s} \\ &= \frac{\widehat{\psi}(2k+s+2)}{\widehat{\mathbb{1}}(2k+2s+2)}z^{k+s}\end{aligned}$$

The lemma is proved by applying $T_{e^{is\theta}\psi}$ to ξ^k n times. ■

We have the following decomposition of $L^2(\mathbb{D}, dA)$ as

$$L^2(\mathbb{D}, dA) = \bigoplus_{k\in\mathbb{Z}} e^{ik\theta}\mathcal{R}$$

where $\mathcal{R}$ is the space of functions on $[0, 1]$ that are square integrable with respect to the measure rdr. Thus every function $f \in L^2(\mathbb{D}, dA)$ has the decomposition

$$f(re^{ik\theta}) = \sum_{k=-\infty}^{+\infty} e^{ik\theta}f_k(r), \qquad f_k \in \mathcal{R}.$$

Moreover, if $f \in L^\infty(\mathbb{D}, dA) \subset L^2(\mathbb{D}, dA)$ then for each $r \in [0, 1)$,

$$\mid f_k(r)\mid = \frac{1}{2\pi}\mid\int_0^{2\pi} f(re^{i\theta})e^{-ik\theta}\,d\theta\mid \le \sup_{z\in\mathbb{D}}\mid f(z)\mid, \quad \forall k \in \mathbb{Z}$$

and so the functions f_k are bounded in the disk.

In [7] we proved the following results which we will use in the proof of our main theorem.

Proposition 4. *Let ϕ be a nonzero bounded radial function, p be a positive integer and $f(re^{ik\theta}) = \sum_{k=-\infty}^{+\infty} e^{ik\theta}f_k(r) \in L^\infty(\mathbb{D}, dA)$. Then*

a) *T_f commutes with $T_{e^{ip\theta}\phi}$ if and only if $T_{e^{ik\theta}f_k}$ commutes with $T_{e^{ip\theta}\phi}$ for all $k \in \mathbb{Z}$.*

b) *If there exists $k \in \mathbb{Z}_-$ and a bounded radial function f_k such that*

$$T_{e^{ip\theta}\phi} T_{e^{ik\theta} f_k} = T_{e^{ik\theta} f_k} T_{e^{ip\theta}\phi}$$

then f_k must be equal to zero.

c) *If there exists $k \in \mathbb{Z}_+$ and a bounded radial function f_k such that*

$$T_{e^{ip\theta}\phi} T_{e^{ik\theta} f_k} = T_{e^{ik\theta} f_k} T_{e^{ip\theta}\phi}$$

then f_k is unique up to a constant factor. In particular f_0 is a constant.

Thus if $p > 0$, $f(re^{ik\theta}) = \sum_{k=-\infty}^{+\infty} e^{ik\theta} f_k(r)$ and T_f commutes with $T_{e^{ip\theta}\phi}$ then each f_k is uniquely determined up to multiplication by a constant and equal to 0 for $k < 0$.
Next we present two technical but easy results which permit us to prove Propositions 7 and 9 the principal results of this section.

Remark 5. *Let $(a_l)_{l\in\mathbb{N}}$ and $(b_l)_{l\in\mathbb{N}}$ be two nonvanishing sequences and p and s two positive integers such that*

$$a_{l+s} b_l = b_{l+p} a_l \quad \text{for all } l \in \mathbb{N}. \tag{2}$$

Then if

$$A_k = \prod_{j=0}^{s-1} a_{k+jp} \quad \text{and} \quad B_k = \prod_{j=0}^{p-1} b_{k+js}$$

we have :

$$A_k B_{k+p} = A_{k+p} B_k \quad \text{for all } k \in \mathbb{N}.$$

(Just multiply the p equations obtained by taking $l = k, k+s, .., k+(p-1)s$ in (2) together to see that, if (2) is true, then

$$\frac{B_{k+p}}{B_k} = \frac{a_{k+ps}}{a_k} = \frac{A_{k+p}}{A_k} \quad \text{for all } k \in \mathbb{N}.)$$

Notation: Let S and T be two functions (resp. two operators). We will say that $S \equiv T$ if there exists a constant $c \neq 0$ such that $S = cT$.

Lemma 6. *Let F and G be two nonzero bounded holomorphic functions on the half plane $\Pi = \{z : \Re z > 2\}$. If there exists $p \in \mathbb{N}$ such that*

$$F(z)G(z+p) = F(z+p)G(z) \text{ for all } z \in \Pi \tag{3}$$

then $F \equiv G$.

PROOF. Suppose that (3) is true. Then, if (as above) we multiply the k equations obtained by taking $z_n = z + np$ for $n = 0, ..., k-1$, we have

$$F(z)G(z+kp) = F(z+kp)G(z) \text{ for all } k \in \mathbb{N}. \tag{4}$$

Now, let $z_0 \in \Pi$ such that $G(z_0) \neq 0$ and let $E = \{k \in \mathbb{N} : G(z_0+kp) = 0\}$. If $\sum_{k\in E} \Re(\frac{1}{|z_0+kp|}) = \infty$, then Theorem 1 implies that $G = 0$. This contradicts the hypothesis of the lemma. Thus $\sum_{k\in E^c} \Re(\frac{1}{|z_0+kp|}) = \infty$ where E^c is the complement in $\mathbb{N}$ of the set E.
Now, equation (4) implies that

$$\frac{F(z_0+kp)}{G(z_0+kp)} = \frac{F(z_0)}{G(z_0)} \text{ for all } k \in E^c.$$

So, applying Theorem 1 to the function $F - cG$ where $c = \frac{F(z_0)}{G(z_0)}$, completes the proof. ∎

Let p and s be two positive integers and ψ a bounded radial function.
If $\left(T_{e^{is\theta}\psi}\right)^p$ is a Toeplitz operator then it is the unique quasihomogeneous Toeplitz operator of degree ps (see Proposition 3 and Proposition 4 of [7]) which commutes with $T_{e^{is\theta}\psi}$. It is natural to ask whether all nonzero Toeplitz operators which are of quasihomogeneous degree a multiple of s and which commute with $T_{e^{is\theta}\psi}$, are of this form.

Proposition 7. *Let p and s be two positive integers and ϕ and ψ be two nonzero bounded radial functions such that*

$$T_{e^{ip\theta}\phi}T_{e^{is\theta}\psi} = T_{e^{is\theta}\psi}T_{e^{ip\theta}\phi}. \tag{5}$$

Then

$$\left(T_{e^{ip\theta}\phi}\right)^s \equiv \left(T_{e^{is\theta}\psi}\right)^p. \tag{6}$$

PROOF. For all $k \in \mathbb{N}$, let

$$a_k = \frac{\widehat{\phi}(2k+p+2)}{\widehat{\mathbb{1}}(2k+2p+2)} \quad \text{and} \quad b_k = \frac{\widehat{\psi}(2k+s+2)}{\widehat{\mathbb{1}}(2k+2s+2)}$$

so that

$$T_{e^{ip\theta}\phi}(\xi^k)(z) = a_k z^{k+p} \quad \text{and} \quad T_{e^{is\theta}\psi}(\xi^k)(z) = b_k z^{k+s}.$$

Then equation (5) shows that $a_{k+s}b_k = b_{k+p}a_k$ for all $k \in \mathbb{Z}_+$ and so Remark 5 implies that

$$\prod_{j=0}^{s-1} a_{k+jp} \prod_{j=0}^{p-1} b_{k+p+js} = \prod_{j=0}^{s-1} a_{k+p+jp} \prod_{j=0}^{p-1} b_{k+js}. \tag{7}$$

Let F and G be the two bounded holomorphic functions defined for all $z \in \Pi$ by

$$F(z) = \prod_{j=0}^{p-1} \widehat{\mathbb{1}}(z+2js+2s) \prod_{j=0}^{s-1} \widehat{\phi}(z+2jp+p)$$

and

$$G(z) = \prod_{j=0}^{s-1} \widehat{\mathbb{1}}(z+2jp+2p) \prod_{j=0}^{p-1} \widehat{\psi}(z+2js+s).$$

Then equation (7) is equivalent to

$$F(2k+2)G(2k+2p+2) = F(2k+2p+2)G(2k+2) \quad \text{for all } k \in \mathbb{Z}_+.$$

Now, applying Theorem 1, in the form of Remark 2, implies that

$$F(z)G(z+2p) = F(z+2p)G(z) \quad \text{for all } z \in \Pi.$$

Finally, using Lemma 6, we obtain that :

$$\frac{\prod_{j=0}^{s-1} \widehat{\phi}(z+2jp+p)}{\prod_{j=0}^{s-1} \widehat{\mathbb{1}}(z+2jp+2p)} \equiv \frac{\prod_{j=0}^{p-1} \widehat{\psi}(z+2js+s)}{\prod_{j=0}^{p-1} \widehat{\mathbb{1}}(z+2js+2s)} \quad \text{for all } z \in \Pi,$$

and Lemma 3 completes the proof. ∎

Remark 8. i) *We will assume that $\left(T_{e^{ip\theta}\phi}\right)^0 = I$ where I is the identity operator of L^2_a onto L^2_a.*

ii) *If p and s are both negative integers and if $T_{e^{ip\theta}\phi}T_{e^{is\theta}\psi} = T_{e^{is\theta}\psi}T_{e^{ip\theta}\phi}$, then by considering the adjoint operators we obtain*

$$T_{e^{-is\theta}\psi}T_{e^{-ip\theta}\phi} = T_{e^{-ip\theta}\phi}T_{e^{-is\theta}\psi}$$

and so Proposition 7 implies that $\big(T_{e^{-ip\theta}\phi}\big)^{-s} \equiv \big(T_{e^{-is\theta}\psi}\big)^{-p}$.
Now, by considering once again the adjoint operators we see that

$$\big(T_{e^{ip\theta}\phi}\big)^{-s} \equiv \big(T_{e^{is\theta}\psi}\big)^{-p}.$$

Proposition 9. *Let ϕ and ψ be two nonzero bounded radial functions and n, p and s be positive integers. Then*

$$\big(T_{e^{ips\theta}\phi}\big)^{n} = \big(T_{e^{is\theta}\psi}\big)^{np} \implies T_{e^{ips\theta}\phi} \equiv \big(T_{e^{is\theta}\psi}\big)^{p}.$$

PROOF. For all $k \in \mathbb{Z}_+$, let

$$a_k = 2(k+ps+1)\widehat{\phi}(2k+ps+2) \quad \text{and} \quad b_k = \prod_{j=0}^{p-1} 2(k+js+s+1)\widehat{\psi}(2k+2js+s+2)$$

so that

$$\big(T_{e^{ips\theta}\phi}\big)^{n} = \big(T_{e^{is\theta}\psi}\big)^{np} \Leftrightarrow \prod_{j=0}^{n-1} a_{k+jps} = \prod_{j=0}^{n-1} b_{k+jps} \quad \text{for all } k \in \mathbb{Z}_+$$

and

$$T_{e^{ips\theta}\phi} = \big(T_{e^{is\theta}\psi}\big)^{p} \Leftrightarrow a_k = b_k \quad \text{for all } k \in \mathbb{Z}_+.$$

Suppose that

$$\prod_{j=0}^{n-1} a_{k+jps} = \prod_{j=0}^{n-1} b_{k+jps} \quad \text{for all } k \in \mathbb{Z}_+. \tag{8}$$

If we multiply the equation (8) and the equation obtained by replacing k by $k+ps$ in the equation (8) together we obtain that

$$a_k b_{k+nps} = a_{k+nps} b_k \quad \text{for all } k \in \mathbb{Z}_+. \tag{9}$$

Now consider two bounded holomorphic functions F and G defined in the right half plane Π by

$$F(z) = \widehat{\phi}(z+ps)\prod_{j=0}^{p-1}\widehat{\mathbb{1}}(z+2js+2s)$$

and

$$G(z) = \widehat{\mathbb{1}}(z+2ps)\prod_{j=0}^{p-1}\widehat{\psi}(z+2js+s).$$

Then equation 9 is equivalent to

$$F(z)G(z+2nps) = F(z+2nps)G(z) \text{ for all } z \in \Pi.$$

Hence, Lemma 6 implies that

$$F(z) \equiv G(z) \text{ for all } z \in \Pi,$$

and Lemma 3 completes the proof. ■

Remark 10. *In* [7]*(Proposition 6) we prove that if $p > 0$ and ϕ is a nonzero bounded radial function and if there exists a bounded radial function ψ such that T_ψ commutes with $T_{e^{ip\theta}\phi}$ then ψ must be a constant. Here is another proof of this proposition. In fact, using Proposition 7, we have $\left(T_\psi\right)^p \equiv I$, so Proposition 9 implies that $T_\psi \equiv I$, and so, that $\psi \equiv \mathbb{1}$ since I is the Toeplitz operator of symbol $\mathbb{1}$.*

4. Main result

Let p be a positive integer. We start this section with the definition of the T-p^{th} root of quasihomogeneous Toeplitz operator of degree p or $-p$. This new notion plays a important role in the remainder of the paper.

Definition 11. *Let ϕ be a nonzero bounded radial function and p be a positive integer. We say that the Toeplitz operator $T_{e^{ip\theta}\phi}$ has a T-p^{th} root $T_{e^{i\theta}\psi}$ if and only if there exists a nonzero bounded radial function ψ such that*

$$T_{e^{ip\theta}\phi} = \left(T_{e^{i\theta}\psi}\right)^p.$$

Remark 12.

i) *The T-p^{th} root of a quasihomogeneous Toeplitz operator is unique. In fact, suppose that $T_{e^{ip\theta}\phi}$ has two T-p^{th} roots $T_{e^{i\theta}\psi}$ and $T_{e^{i\theta}\widetilde{\psi}}$ then $\left(T_{e^{i\theta}\psi}\right)^p = \left(T_{e^{i\theta}\widetilde{\psi}}\right)^p$. Then, by Proposition 9, we have that $T_{e^{i\theta}\psi} = T_{e^{i\theta}\widetilde{\psi}}$ which implies that $\psi = \widetilde{\psi}$.*

ii) *If the quasihomogeneous degree is negative we have an analogous definition of the T-p^{th} root. Let p be a positive integer and ϕ be a bounded radial function. Then, we say that $T_{e^{-ip\theta}\phi}$ has a T-p^{th} root if there exists a bounded radial function ψ such that $T_{e^{-ip\theta}\phi} = \left(T_{e^{-i\theta}\psi}\right)^p$. It is easy to see, by taking adjoints, that $T_{e^{-ip\theta}\phi}$ has a T-p^{th} root $T_{e^{-i\theta}\psi}$ if and only if $T_{e^{ip\theta}\phi}$ has a T-p^{th} root $T_{e^{i\theta}\psi}$.*

Examples :

i) $T_{e^{i\theta}(\frac{r+r^5}{2})}$ is the T-2^{th} root of $T_{e^{2i\theta}r^6}$.

ii) $T_{e^{i\theta}(\frac{3r+2r^5+3r^9}{8})}$ is the T-2^{th} root of $T_{e^{2i\theta}r^{10}}$.

Now, if $T_{e^{i\theta}\psi}$ is the T-p^{th} root of $T_{ip\theta}\phi$ and if $\left(T_{e^{i\theta}\psi}\right)^k$ (for k in $\mathbb{N}$) is a Toeplitz operator, then $\left(T_{e^{i\theta}\psi}\right)^k$ is the unique nonzero quasihomogeneous Toeplitz operator of degree k which can commute with $T_{e^{ip\theta}\phi}$. What we prove below is that if $T_{e^{ip\theta}\phi}$ has a T-p^{th} root $T_{e^{i\theta}\psi}$, then the *only* nonzero quasihomogeneous Toeplitz operator of degree s which commutes with $T_{e^{ip\theta}\phi}$ is a s^{th} power of $T_{e^{i\theta}\psi}$, extending the result (Propositions 7 and 9) of section 3 in this case.

Theorem 13. *Let ϕ be a nonzero bounded radial function and p be a positive integer. Assume that $T_{e^{ip\theta}\phi}$ has a T-p^{th} root $T_{e^{i\theta}\psi}$. Suppose that*

$$f(re^{i\theta}) = \sum_{k=-\infty}^{+\infty} e^{ik\theta} f_k(r) \in L^\infty(\mathbb{D}, dA)$$

is such that

$$T_f T_{e^{ip\theta}\phi} = T_{e^{ip\theta}\phi} T_f. \tag{10}$$

Then

i) *$f_k = 0$ for $k < 0$.*

ii) *If $k \geq 0$ and $\left(T_{e^{i\theta}\psi}\right)^k$ is a Toeplitz operator, then either $T_{e^{ik\theta}f_k} \equiv \left(T_{e^{i\theta}\psi}\right)^k$ or $f_k = 0$.*

iii) *If $k \geq 0$ and $\left(T_{e^{i\theta}\psi}\right)^k$ is not a Toeplitz operator, then $f_k = 0$.*

PROOF. Assertion a) of Proposition 4 implies that if equation (10) is true, then

$$T_{e^{ik\theta}f_k} T_{e^{ip\theta}\phi} = T_{e^{ip\theta}\phi} T_{e^{ik\theta}f_k}, \text{ for all } k \in \mathbb{Z}.$$

Thus i) is a direct consequence of assertion b) of Proposition 4.
Now, to prove ii), let k be a positive integer such that $\left(T_{e^{i\theta}\psi}\right)^k$ is a Toeplitz operator. Then $\left(T_{e^{i\theta}\psi}\right)^k$ is a quasihomogeneous Toeplitz operator of degree k which commutes with $T_{e^{ip\theta}\phi}$. So, if f_k is not identically equal to zero, then f_k is a bounded nonzero radial function such that $T_{e^{ik\theta}f_k}$ commutes with $T_{e^{ip\theta}\phi}$. Thus, assertion c) of Proposition 4 implies that $T_{e^{ik\theta}f_k} \equiv \left(T_{e^{i\theta}\psi}\right)^k$.
Finally, let k be a positive integer such that $\left(T_{e^{i\theta}\psi}\right)^k$ is not a Toeplitz operator and suppose that there exists a nonzero bounded radial function f_k such that $T_{e^{ik\theta}f_k}$ commutes with $T_{e^{ip\theta}\phi}$. Then Proposition 7 implies that

$$\left(T_{e^{ik\theta}f_k}\right)^p \equiv \left(T_{e^{ip\theta}\phi}\right)^k.$$

Thus $\left(T_{e^{ik\theta}f_k}\right)^p \equiv \left(T_{e^{i\theta}\psi}\right)^{kp}$ and Proposition 9 implies that $T_{e^{ik\theta}f_k} \equiv \left(T_{e^{i\theta}\psi}\right)^k$ which contradicts our hypothesis. This proves iii). ■

Before starting with corollaries, we state an interesting theorem which follows from [5] and give an idea of its proof. In fact we will apply this theorem to see that if p is any positive integer and m is any nonnegative integer then the Toeplitz operator $T_{e^{ip\theta}r^m}$ always has a T-p^{th} root.

Theorem 14. *Let $p \geq 1$ and $m \geq 0$ be two integers. For all integers s, such that $1 \leq s < p$, there exists a unique bounded radial function ψ such that*

$$T_{e^{is\theta}\psi} T_{e^{ip\theta}r^m} = T_{e^{ip\theta}r^m} T_{e^{is\theta}\psi}. \tag{11}$$

PROOF. (This is a slight variation of the proof found in [5])
If $m \geq 0, p \geq 1$ and $1 \leq s < p$, we define the radial functions f and g by

$$f(r) = 2pr^{2s}(1-r^{2p})^{-\frac{s}{p}} \quad \text{and} \quad g(r) = 2pr^{m+p}(1-r^{2p})^{\frac{s}{p}-1}.$$

Let ψ be the radial function defined by

$$r^s\psi = f *_M g.$$

Čučković and Rao prove, using a long rather technical calculation, that ψ is bounded. Here, we will show that ψ satisfies (11). To do this, we need only verify that for $k \in \mathbb{Z}_+$:

$$\frac{2k+2p+2}{2k+m+p+2}\widehat{r^s\psi}(2k+2p+2) = \frac{2k+2s+2}{2k+m+p+2s+2}\widehat{r^s\psi}(2k+2).$$

By (1), we have $\widehat{r^s\psi}(2k+2) = \widehat{f}(2k+2)\widehat{g}(2k+2)$. A simple substitution $t = r^{2p}$ shows that

$$\widehat{f}(2k+2) = B\Big(\frac{2k+2s+2}{2p}, 1-\frac{s}{p}\Big) \quad \text{and} \quad \widehat{g}(2k+2) = B\Big(\frac{2k+m+p+2}{2p}, \frac{s}{p}\Big)$$

where B denotes the beta function. Using the well-known identities $B(z_1, z_2) = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)}$ and $\Gamma(1+z) = z\Gamma(z)$, where Γ is the gamma function, it is easy to see that

$$\widehat{r^s\psi}(2k+2p+2) = \frac{(2k+2s+2)(2k+m+p+2)}{(2k+2p+2)(2k+m+p+2s+2)}\widehat{r^s\psi}(2k+2) \tag{12}$$

which finishes the proof. ■

Remark 15. i) *It is trivial that $T_{e^{ip\theta}r^m}$ commutes with itself. So, if $p = s$, assertion c) of Proposition 4 implies that $\psi \equiv r^m$.*

ii) *We wish to highlight the following case. If $m = (2n+1)p$ for $n \in \mathbb{N}$ then the function ψ exists for all $s \in \mathbb{N}$. In fact, if we substitute $m = (2n+1)p$ in (12) and use Theorem 1, we obtain for all $z \in \Pi$*

$$\frac{\widehat{r^s\psi}(z+2p)}{\widehat{r^s\psi}(z)} = \frac{F(z+2p)}{F(z)}, \text{ where } F(z) = \frac{\Gamma(\frac{z+2s}{2p})\Gamma(\frac{z}{2p}+n+1)}{\Gamma(\frac{z}{2p}+1)\Gamma(\frac{z+2s}{2p}+n+1)}.$$

Now, using the identity $\Gamma(1+z) = z\Gamma(z)$ repeatedly, we have

$$F(z) = 2p\frac{\prod_{j=0}^{n-1}(z+2jp+2p)}{\prod_{j=0}^{n}(z+2jp+2s)}$$

which is a proper fraction in z and can be written as

$$F(z) = \sum_{j=0}^{n} \frac{a_j}{z+2jp+2s}. \tag{13}$$

Since $\frac{1}{z+2jp+2s} = \widehat{r^{2jp+2s}}(z)$, it follows by Lemma 6 that

$$\widehat{r^s\psi}(z) \equiv \sum_{j=0}^{n} a_j \widehat{r^{2jp+2s}}(z)$$

where the a_j are defined by (13), and so Theorem 1 implies that

$$\psi(r) \equiv \sum_{j=0}^{n} a_j r^{2jp+s}.$$

Next, we give some easy but interesting consequences of Theorem 14.

Corollary 16. *For all integers $m \geq 0$, $p \geq 1$, and $s \geq 1$ there exists a bounded radial function ψ such that $\left(T_{e^{is\theta}\psi}\right)^p \equiv T_{e^{ips\theta}r^m}$.*

PROOF. Let $m \geq 0$, $p \geq 1$, and $s \geq 1$ be integers. Theorem 14 implies that there exists a bounded radial function ψ such that

$$T_{e^{is\theta}\psi}T_{e^{ips\theta}r^m} = T_{e^{ips\theta}r^m}T_{e^{is\theta}\psi}.$$

Using Proposition 7 we have $\left(T_{e^{is\theta}\psi}\right)^{ps} \equiv \left(T_{e^{ips\theta}r^m}\right)^s$ and so, an application of Proposition 9 finishes the proof. ■

In [4], Brown and Halmos studied multiplicativity of Toeplitz operators on the Hardy space and showed that the product of two Toeplitz operators T_f and T_g is equal to a third Toeplitz operator T_h for some f, g and h in $L^\infty(\mathbb{T})$ if and only if f is conjugate analytic or g is analytic -that is, hardly ever. The question of when the product of two Toeplitz operators on the Bergman space is equal to a third is much more complicated and still open. Most work on this question shows that it is not often true that the product of two Toeplitz operators is a Toeplitz operator (see [1] and [6]). But, below, we show that, for certain nontrivial Toeplitz operators $T_{e^{i\theta}\psi}$, not only is $\left(T_{e^{i\theta}\psi}\right)^2$ equal to a Toeplitz operator, but there exists a positive integer k such that $\left(T_{e^{i\theta}\psi}\right)^i$ is a Toeplitz operator for all positive integers $i \leq k$.

Corollary 17. *Let $m \geq 0$ and $p \geq 1$ be two integers. If $T_{e^{ip\theta}r^m}$ has a T-p^{th} root $T_{e^{i\theta}\psi}$ then, for all integers k with $1 \leq k \leq p$, the product $\left(T_{e^{i\theta}\psi}\right)^k$ is a Toeplitz operator.*

PROOF. Let k be an integer such that $1 \leq k \leq p$. By Theorem 14 we know that there exists a bounded radial function ϕ such that $T_{e^{ik\theta}\phi}$ commutes with $T_{e^{ip\theta}r^m}$. So, Proposition 7 implies that

$$\left(T_{e^{ik\theta}\phi}\right)^p \equiv \left(T_{e^{ip\theta}r^m}\right)^k.$$

Thus $\left(T_{e^{ik\theta}\phi}\right)^p = \left(T_{e^{i\theta}\psi}\right)^{kp}$ since $T_{e^{i\theta}\psi}$ is the T-p^{th} root of $T_{e^{ip\theta}r^m}$. And so Proposition 9 finishes the proof. ∎

It is easily seen that if f is a bounded analytic function on $\mathbb{D}$, then T_f is just a multiplication operator. Thus for any integer $k \geq 1$, it is clear that $\left(T_f\right)^k$ is a Toeplitz operator of symbol f^k. By taking adjoints, we can see that the powers of a Toeplitz operator with conjugate analytic symbol is also a Toeplitz operator. These are the trivial cases. The next corollary says there are nontrivial symbols f such that $\left(T_f\right)^k$ is always a Toeplitz operator for all integers $k \geq 1$.

Corollary 18. *There exist bounded radial functions ψ such that for all integers $k \geq 1$ the product $\left(T_{e^{i\theta}\psi}\right)^k$ is still a Toeplitz operator.*

PROOF. Let $n \geq 0$, and $p \geq 1$ be two integers. By Theorem 14 we know that the Toeplitz operator $T_{e^{ip\theta}r^{(2n+1)p}}$ has a T-p^{th} root $T_{e^{i\theta}\psi}$ where ψ is a bounded radial function. Moreover the assertion *ii*) of Remark 15 tells us that, for all integers $k \geq 1$, there exists a bounded radial function ψ_k such that $T_{e^{ik\theta}\psi_k}$ commutes with $T_{e^{ip\theta}r^{(2n+1)p}}$. Thus Proposition 7 implies that $\left(T_{e^{ik\theta}\psi_k}\right)^p \equiv \left(T_{e^{i\theta}\psi}\right)^{kp}$ since $T_{e^{ip\theta}r^{(2n+1)p}} = \left(T_{e^{i\theta}\psi}\right)^p$ and, again, Proposition 9 finishes the proof. ∎

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