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# Products of Toeplitz operators on the Bergman space

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**Abstract.** In 1962 Brown and Halmos gave simple conditions for the product of two Toeplitz operators on Hardy space to be equal to a Toeplitz operator. Recently, Ahern and Čučković showed that a similar result holds for Toeplitz operators with bounded harmonic symbols on Bergman space. For general symbols, the situation is much more complicated. We give necessary and sufficient conditions for the product to be a Toeplitz operator (Theorem 6.1), an explicit formula for the symbol of the product in certain cases (Theorem 6.4), and then show that almost anything can happen (Theorem 6.7).

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## 1. Preliminaries

Let $dA$ denote Lebesgue area measure on the unit disc $\mathbb{D}$, normalized so that the measure of $\mathbb{D}$ equals 1. The Bergman space $L_a^2$ is the Hilbert space consisting of the analytic functions which are contained in $L^2(\mathbb{D}, dA)$. It is well known that $L_a^2$ is a closed subspace of the Hilbert space $L^2(\mathbb{D}, dA)$ and that, for each $z \in \mathbb{D}$, the application:

$$\begin{array}{rcl} L_z : L_a^2 & \longrightarrow & \mathbb{C} \\ f & \mapsto & f(z) \end{array}$$

is continuous and can be represented as $L_z(f) =< f, k_z >$, where:

$$k_z(w) = \frac{1}{(1 - w\overline{z})^2} = \sum_{j=0}^{\infty}(1 + j)w^j\overline{z}^j.$$

This means that, if $P$ is the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto $L^2_a$, then $P$ can be defined by:

$$(Pf)(z) =< f, k_z >= \int_{\mathbb{D}} f(w)\overline{k_z(w)}\, dA(w)$$

For $u \in L^\infty(\mathbb{D}, dA)$, we define the Toeplitz operator with symbol $u$, $T_u : L^2_a \longrightarrow L^2_a$ by the equation:

$$T_u(f)(z) = P(uf)(z) = \int_{\mathbb{D}} u(w)f(w)\overline{k_z(w)}dA(w) \quad (1)$$

The operators defined in this way are the simplest and most natural Toeplitz operators (since the product of an $L^\infty$ and an $L^2$ fonction is always a well defined element of $L^2$). But, for reasons which will become evident, we prefer to consider a more general class of Toeplitz operators.

Let $u$ be any finite complex measure on $\mathbb{D}$. In analogy with equation (1) we can define an operator $T_u$ on $L^2_a$ by:

$$T_u f(z) = \int_{\mathbb{D}} f(w)\overline{k_z(w)}\, du(w) \quad (2)$$

If $du(z) = F(z)dA(z)$ for some $F \in L^1(\mathbb{D}, dA)$, then we simply write $T_u = T_F$. This operator is always defined on the polynomials and the image of any polynomial is always an analytic function on the disc. We are interested in the case where this densely defined operator is bounded in the $L^2_a$ norm. This happens often. For example, if $u$ has compact support, then $T_u$ is not only bounded, but compact. Thus, if $F \in L^1(\mathbb{D}, dA)$ and there is an $r \in (0, 1)$ such that $F$ is (essentially) bounded on the annulus $\{z : r < |z| < 1\}$ then $F$ is equal to the sum of an $L^1$ function with compact support and an $L^\infty$ function and so $T_F$ is a bounded operator. There is, unfortunately, no characterization of the functions in $L^1(\mathbb{D}, dA)$ which correspond to bounded operators. This motivates two of the following definitions.

**Definition 1.1.** *Let* $F \in L^1(\mathbb{D}, dA)$.

*(a) We say that* $F$ *is a T-function if the equation (1), with* $u = F$*, defines a bounded operator on* $L^2_a$.

*(b) If* $F$ *is a T-function, we write* $T_F$ *for the continuous extension of the operator defined by equation (1). We say that* $T_F$ *is a Toeplitz operator if and only if* $T_F$ *is defined in this way.*

*(c) If there is an* $r \in (0, 1)$ *such that* $F$ *is (essentially) bounded on the annulus* $\{z : r < |z| < 1\}$ *then we say that* $F$ *is "nearly bounded".*

Notice that the T-functions form a proper subset of $L^1(\mathbb{D}, dA)$ which contains all bounded and 'nearly bounded' functions.

## 2. History and motivation

The question to be considered in this article is: When is the product of two Toeplitz operators $T_f$ and $T_g$ equal to a Toeplitz operator $T_h$? The corresponding question for Toeplitz operators on the Hardy space was elegantly and simply resolved by Brown and Halmos in 1964. Let $\Gamma$ be the unit circle in the complex plane and let $H^2$ be the Hardy space on the unit disc $\mathbb{D}$. As, usual, for $f$ in $L^\infty(\Gamma)$ we define the Toeplitz operator $T_f$ by the equation

$$T_f(\phi) = P^{H^2}(f\phi)$$

where $P^{H^2}$ is orthogonal projection from $L^2(\Gamma)$ onto $H^2$. In this case, even a definition in terms of the reproducing kernel, as in (1), does not give any other Toeplitz operators. We say that a function in $L^\infty(\Gamma)$ is analytic if all of it's negative Fourier coefficients are equal to 0. Brown and Halmos show in [7] that, for $f$ and $g$ any two functions in $L^\infty(\Gamma)$. $T_f T_g = T_h$ if and only if either
(a) $g$ is analytic
or
(b) $\overline{f}$ is analytic. They also show that, in both cases $h = fg$. The sufficiency of these conditions is 'obvious' since:
(1) If $g$ is analytic, then $T_g(\phi) = g\phi$.
(2) For any $\psi \in L^\infty$ , $T_\psi^* = T_{\overline{\psi}}$.

In the Bergman space, as usual, things are much more complicated. Conditions (a) and (b) are still sufficient - since (1) and (2) are still true - but they are no longer necessary. Two papers on the subject have appeared recently ([1], [2]). In [2] the authors get a Brown-Halmos type result. They show that conditions (a) and (b) above are both necessary and sufficient under the assumptions that $f$, $g$ and $h$ are bounded harmonic functions and that $\widetilde{\Delta}h = (1-|z|^2)^2\Delta h$ is bounded. More generally, in [1], Ahern considers the product $T_f T_g$ for $f$ and $g$ bounded harmonic functions on the disc such that $f = f_1 + \overline{f_2}$ and $g = g_1 + \overline{g_2}$ with $f_1$,$f_2$,$g_1$ and $g_2$ are bounded analytic functions. He shows that $T_f T_g$ is a Toeplitz operator $T_\psi$ if and only if there exist $p$ and $q$ holomorphic polynomials with degree of $pq$ less than or equal to 3 such that $f_1 = p \circ \phi_a$ and $g_2 = q \circ \phi_a$ where $\phi_a$ is the automorphism of $\mathbb{D}$ defined by

$$\phi_a(z) = \frac{a-z}{1-\overline{a}z} \qquad (z \in \mathbb{D}).$$

He also shows that, if $f_1$ or $g_2$ is not equal to zero, then $\psi \neq fg$.

In this article we discuss the question for more general symbols. We find necessary and sufficient conditions for the product of certain symbols to be a Toeplitz operator and give a formula for the symbol of the product. Much work remains to be done, both in resolving the question for operators with completely arbitrary symbols and in getting a more precise description of a 'T-function'.

## 3. The Mellin transform and Mellin convolution

One of our most useful tools in the following calculations will be the Mellin transform (closely related, using the change of variables $s = e^{-u}$, to the Laplace transform).

The Mellin transform $\widehat{\varphi}$ of a function $\varphi$ is defined by the equation:

$$\widehat{\varphi}(z) = \int_0^\infty \varphi(s) s^{z-1}\, ds.$$

We shall apply the Mellin transform to functions in $L^1([0,1], rdr)$ (considered to be equal to zero on the interval $]1,\infty[$). It is clear that, for these functions, the Mellin transform is (well) defined on $\{z : Rez \geq 2\}$ and analytic on $\{z : \mathrm{Re} z > 2\}$. It is important that a function is determined by the value of a certain number of its Mellin coefficients. This following lemma is proved in [9].

**Lemma 3.1.** *Let $\varphi \in L^1([0,1], rdr)$. If there exist $n_0, p \in \mathbb{N}$ such that:*

$$\widehat{\varphi}(n_0 + pk) = 0 \qquad \text{for all} \qquad k \in \mathbb{N}$$

*then $\varphi = 0$.*

When considering the product of two Toeplitz operators we shall often be confronted with the “Mellin” or “multiplicative” convolution of their symbols. We denote the Mellin convolution of two functions $f$ and $g$ by $f *_M g$ and we define it to be:

$$(f *_M g)(r) = \int_r^1 f(\frac{r}{t}) g(t)\, \frac{dt}{t}.$$

The multiplication $*_M$ is related to the normal convolution by the change of variables discussed above.

It is easy to see that the Mellin transform converts the convolution product into a pointwise product, i.e that:

$$(\widehat{f *_M g})(r) = \widehat{f}(r)\widehat{g}(r)$$

and that, if $f$ and $g$ are in $L^1([0,1], rdr)$ then so is $f *_M g$.

## 4. Products of Toeplitz operators with radial symbols

Let $\varphi \in L^1(\mathbb{D}, dA)$ be a radial function, i.e. suppose that:

$$\varphi(z) = \varphi(|z|) \qquad (z \in \mathbb{D}).$$

Then, if $\varphi$ is a T-function, the Toeplitz operator with symbol $\varphi$ acts in a very simple way. In fact, if we define the function $\varphi_r$ on $[0,1]$ by

$$\varphi_r(s) = \varphi(s)$$

then a direct calculation shows that:

$$< T_\varphi(z^k), z^l >= \begin{cases} 0 & \text{for } k \neq l \\ 2\widehat{\varphi_r}(2k+2) & \text{for } k = l \end{cases} \tag{3}$$

so that, if $k \in \mathbb{N}$:

$$T_\varphi(z^k) = (2k+2)\widehat{\varphi_r}(2k+2)z^k. \tag{4}$$

Thus $T_\varphi$ is a diagonal operator on $L^2_a$ with coefficient sequence

$$\big((2k+2)\widehat{\varphi_r}(2k+2)\big)_{k=0}^{\infty}.$$

This makes it relatively simple to work with the product of two operators with such radial symbols.

**Remark:**In the following, we shall often identify an integrable radial function $\varphi$ on the unit disc with the corresponding function $\varphi_r$ defined on the interval $[0,1]$ . For example, if we speak of the multiplicative convolution $*_M$ of two radial functions $\varphi_1$ and $\varphi_2$ , we mean the radial function $\varphi_3$ such that $\varphi_{3,r} = \varphi_{1,r} *_M \varphi_{2,r}$. Similarly, the Mellin coefficients of an integrable radial function $\varphi$ are defined to be those of the function $\varphi_r$.

Now, we define the "radialization" of a function $f \in L^1(\mathbb{D}, dA)$ by:

$$rad(f)(z) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it}z)dt.$$

It is clear that a function $f$ is radial if and only if $rad(f) = f$. This permits us to prove a very simple but essential proposition.

**Proposition 4.1.** *Let $\varphi \in L^1(\mathbb{D}, dA)$. Then the following assertions are equivalent:*
*(a) For all $k \geqslant 0$ there exist $\lambda_k \in \mathbb{C}$ such that $T_\varphi(z^k) = \lambda_k z^k$.*
*(b) $\varphi$ is a radial function.*

**Proof:** Writing out the integrals and changing the order of integration, we see that, for each $n, m \in \mathbb{Z}_+$:

$$\begin{aligned} < T_{rad(\varphi)}z^n, z^m > &= \frac{1}{2\pi}\left(\int_0^{2\pi} e^{i(m-n)t}dt\right) < T_\varphi z^n, z^m > \\ &= \begin{cases} < T_\varphi z^n, z^m > & \text{for } n = m \\ 0 & \text{for } n \neq m \end{cases} \end{aligned}$$

Thus $T_{rad(\varphi)} = T_\varphi$ if and only if (a) is true. And $T_{rad(\varphi)} = T_\varphi$ if and only if $rad(\varphi) = \varphi$.

**Corollary 4.2.** *Let $\varphi_1$ and $\varphi_2$ be radial T-functions. If $T_{\varphi_1}T_{\varphi_2} = T_\psi$ then $\psi$ is a radial T-function.*

**Proof:** Using equation (4) to calculate $T_{\varphi_1}T_{\varphi_2}(z^k)$ we see that Proposition 4.1 implies that $\psi$ is a radial function. Moreover, $T_\psi$ is clearly a bounded operator.

We are now ready to answer the question: when is the product of two Toeplitz operator with radial symbols equal to a Toeplitz operator? The answer to this question is a consequence of our main theorem but we state it separately here to motivate our other calculations.

**Proposition 4.3.** *Let $\varphi_1$ and $\varphi_2$ be radial T-functions. Then $T_{\varphi_1}T_{\varphi_2}$ is equal to the Toeplitz operator $T_\psi$ if and only if $\psi$ is a solution of the equation:*

$$\mathbb{1} *_M \psi = \varphi_1 *_M \varphi_2 \tag{5}$$

**Proof:** By direct calculation

$$T_{\varphi_1}T_{\varphi_2}(z^k) = T_\psi(z^k) \text{ for } k \geqslant 0$$

if and only if

$$\frac{1}{2k+2}\widehat{\psi}(2k+2) = \widehat{\varphi_1 *_M \varphi_2}(2k+2) \tag{6}$$

But, using Lemma 3.1, equation (5) is equivalent to equation (6) since:

$$\widehat{\mathbb{1}}(2k+2) = \widehat{\chi_{[0,1]}}(2k+2) = \frac{1}{2k+2}$$

One can now have fun calculating lots of products of Toeplitz operators. For example:

$$T_{|z|^n}T_{|z|^m} = \begin{cases} \frac{n}{n-m}T_{|z|^n} - \frac{m}{n-m}T_{|z|^m} & n \neq m \\ T_{|z|^n(1+n\log|z|)} & n = m \end{cases}$$

## 5. Products of Toeplitz operators with quasihomogeneous symbols

Let $\mathcal{R}$ be the space of square integrable radial functions on $\mathbb{D}$. As before, we identify these functions with the associated functions on $[0,1]$ that are square integrable with respect to $rdr$ measure. By using that trigonometric polynomials are dense in $L^2(\mathbb{D}, dA)$ and that, for $k_1 \neq k_2$, $e^{ik_1\theta}\mathcal{R}$ is orthogonal to $e^{ik_2\theta}\mathcal{R}$ we see that:

$$L^2(\mathbb{D}, dA) = \bigoplus_{k\in\mathbb{Z}} e^{ik\theta}\mathcal{R}.$$

Even though this type of decomposition does not exist for $L^1(\mathbb{D}, dA)$ (see [12]), we feel that "STEP1" is to study products of Toeplitz operators with symbols in subspaces of the form $e^{ik\theta}$·radial functions.

**Definition 5.1.** *Let $\varphi$ be a function in $L^1(\mathbb{D}, dA)$ which is of the form $e^{ik\theta} \cdot f$ where $f$ is a radial function. Then we say that $\varphi$ is a quasihomogeneous function of quasihomogeneous degree $k$.*

The third author used Definition 5.1 in her analysis of finite rank Hankel operators on the harmonic Bergman space [13].

**Proposition 5.2.** *Let $k_1$ and $k_2$ be greater than or equal to zero and let $\varphi_1$ and $\varphi_2$ be quasihomogeneous T-functions in $L^1(\mathbb{D}, dA)$ of quasihomogeneous degrees $k_1$ and $-k_2$ respectively. If there exists a T-function $\psi$ such that*

$$T_{\varphi_1}T_{\varphi_2} = T_\psi$$

*then $\psi$ is of quasihomogeneous degree $k_1 - k_2$.*

**Proof:** Let $\varphi_1$, $\varphi_2$ and $\psi$ be as above and let $f_1$ and $f_2$ be radial functions such that

$$\varphi_1 = e^{ik_1\theta} f_1 \text{ and } \varphi_2 = e^{-ik_2\theta} f_2$$

As discussed in Section 2, if $f$ is antianalytic or if $g$ is analytic then $T_f T_g = T_{fg}$. Thus, since $T_{\varphi_1} T_{\varphi_2} = T_\psi$, we see that:

$$T_{\overline{z}^{k_1}\varphi_1} T_{z^{k_2}\varphi_2} = T_{r^{k_1} f_1} T_{r^{k_2} f_2} = T_{\overline{z}^{k_1} z^{k_2}\psi}$$

Now, by Corollary 4.2, $\overline{z}^{k_1} z^{k_2}\psi$ is a radial function. This shows that $\psi$ is a quasihomogeneous function of quasihomogeneous degree $k_1 - k_2$.

We note that the Prop 5.2 is, in fact, true for any integers $k_1$ and $k_2$ (see [10] ).

A direct calculation gives the following lemma which we shall use often.

**Lemma 5.3.** *Let $k, p \in \mathbb{Z}_+$ and let $\varphi$ be an integrable radial function. Then, if $e^{ip\theta}\varphi$ is a T-function we have*

$$T_{e^{ip\theta}\varphi}(z^k) = 2(k+p+1)\widehat{\varphi}(2k+p+2) z^{k+p}$$

*and*

$$T_{e^{-ip\theta}\varphi}(z^k) = \begin{cases} 0 & \text{if } 0 \leqslant k \leqslant p-1 \\ 2(k-p+1)\widehat{\varphi}(2k-p+2) z^{k-p} & \text{if } k \geqslant p. \end{cases}$$

## 6. Principal Results

We now apply our methods of calculation to the problem of determining whether the product of two Toeplitz operators with quasihomogeneous symbols is equal to a Toeplitz operator.

**Theorem 6.1.** *Let $p, s \in \mathbb{Z}_+$, $p \geq s$ and let $\varphi_1$ and $\varphi_2$ be two integrable radial functions on $\mathbb{D}$ such that $e^{ip\theta}\varphi_1$ and $e^{-is\theta}\varphi_2$ are T-functions. Then*

$$T_{e^{ip\theta}\varphi_1} T_{e^{-is\theta}\varphi_2}$$

*is equal to a Toeplitz operator if and only if there exists an integrable radial function $\psi$ such that*

*(a) $e^{i(p-s)\theta}\psi$ is a T-function;*

*(b) $\widehat{\psi}(2k+p-s+2) = 0$ if $0 \leq k \leq s-1$;*

*(c) $\psi$ is a solution to the equation*

$$\mathbb{1} *_M r^{p+s}\psi = r^p\varphi_1 *_M r^s\varphi_2.$$

*In this case:*

$$T_{e^{ip\theta}\varphi_1} T_{e^{-is\theta}\varphi_2} = T_{e^{i(p-s)\theta}\psi}.$$

**Proof:** Using Proposition 5.2 and Lemma 5.3, one sees that, if $T_{e^{ip\theta}\varphi_1}T_{e^{-is\theta}\varphi_2}$ is a Toeplitz operator, then this operator is of the form $T_{e^{i(p-s)\theta}\psi}$, with $e^{i(p-s)\theta}\psi$ a T-function and:

$$\widehat{\psi}(2k+p-s+2) = \begin{cases} 0 & \text{if } 0 \leq k < s \\ 2(k-s+1)\widehat{\varphi_1}(2k+p-2s+2)\widehat{\varphi_2}(2k-s+2) & \text{if } k \geq s. \end{cases}$$

Thus, (b) is true, and, for $k \geq s$:

$$\frac{\widehat{r^{p+s}\psi}(2(k-s)+2)}{2(k-s)+2} = \widehat{r^p\varphi_1}(2(k-s)+2)\widehat{r^s\varphi_1}(2(k-s)+2). \tag{7}$$

Now the same reasoning as in the proof of Proposition 4.3 shows that equation (7) is equivalent to condition (c).

Conversely if $e^{i(p-s)\theta}\psi$ is a T-function and $\psi$ satisfies (a), (b), and (c) then $T_{e^{i(p-s)\theta}\psi}$ is a bounded Toeplitz operator taking the same values on the analytic polynomials as the product $T_{e^{ip\theta}\varphi_1}T_{e^{-is\theta}\varphi_2}$. This completes the proof.

**Remark 6.2.** *Notice that the case $0 \leqslant p < s$ is also covered by the theorem above since $T_{e^{ip\theta}\varphi_1}T_{e^{-is\theta}\varphi_2}$ is equal to a Toeplitz operator (with symbol $\varphi_3$) if and only if its adjoint $T_{e^{is\theta}\overline{\varphi_2}}T_{e^{-ip\theta}\overline{\varphi_1}}$ is equal to a Toeplitz operator (with symbol $\overline{\varphi_3}$).*

One can also obtain complicated results concerning linear combinations of quasihomogeneous symbols, none of which seem worth stating explicitly. By applying the unitary operator

$$\begin{aligned} U_w : \quad & L_a^2 \to L_a^2 \\ & f \longmapsto U_w f(z) = (f \circ \Phi_w)(z)\Phi_w'(z) \end{aligned}$$

where $\Phi_w(z) = \frac{z-w}{1-\overline{w}z}$ is the automorphism of the unit disc sending $w$ to 0; one obtains a generalization of Theorem 6.1 to several other families of symbols.

**Corollary 6.3.** *Let $p \geqslant s$ and let $\varphi_1$, $\varphi_2$ and $\psi$ be as in Theorem 6.1. If*

$$\widetilde{\varphi}_1 = (e^{ip\theta}\varphi_1) \circ \Phi_w, \quad \widetilde{\varphi}_2 = (e^{-is\theta}\varphi_2) \circ \Phi_w \text{ and } \widetilde{\psi} = (e^{i(p-s)\theta}\psi) \circ \Phi_w$$

*then the product of the Toeplitz operators $T_{\widetilde{\varphi}_1}T_{\widetilde{\varphi}_2}$ is equal to the Toeplitz operator $T_{\widetilde{\psi}}$.*

**Proof:** This is an immediate consequence of Theorem 6.1 and the classic result (see [5] for example) that, if $T_f$ is a Toeplitz operator then $U_w^{-1}T_fU_w = T_{f\circ\Phi_w}$.

Now, suppose that $\varphi_1$ and $\varphi_2$ are radial functions such that the function $r^p\varphi_1 *_M r^s\varphi_2$ is differentiable on the interval (0,1) (when interpreted as a function of $r$). Then the convolution equation in Theorem 6.1 is easy to solve.

**Theorem 6.4.** *Let $p, s \in \mathbb{Z}_+, p \geqslant s$ and let $\varphi_1$ and $\varphi_2$ be integrable radial functions such that the function $\Lambda$ defined by*

$$\Lambda(r) = r^p\varphi_1(r) *_M r^s\varphi_2(r)$$

*is almost everywhere differentiable on (0,1). Let $\psi$ be the radial function associated with the function*

$$\psi_r(t) = -t^{1-(p+s)}\Lambda^{'}(t)$$

*defined on the interval $[0,1)$. Then the product $T_{e^{ip\theta}\varphi_1}T_{e^{-is\theta}\varphi_2}$ is equal to the Toeplitz operator $T_{e^{i(p-s)\theta}\psi}$ if and only if*

(i) *The function $e^{i(p-s)\theta}\psi$ is a T-function.*

(ii) *$\widehat{\psi}(2k+p-s+2) = 0$ for $0 \leqslant k \leqslant s-1$.*

**Proof:** $\psi$ is a solution of the equation

$$\mathbb{1} *_M r^{p+s}\psi = \Lambda$$

if and only if

$$\int_t^1 r^{p+s-1}\psi(r)\,dr = \Lambda(t).$$

By differentiating both sides, we see that this means that

$$\psi(t) = -t^{1-(p+s)}\Lambda^{'}(t).$$

Next, an easy but interesting application of Theorem 6.4.

**Corollary 6.5.** *Let $p \geq s$ with $p, s \in \mathbb{Z}_+$, and let $l_1$ and $l_2$ be two real numbers greater than or equal to -1. Then the product*

$$T_{e^{ip\theta}|z|^{l_1}}T_{e^{-is\theta}|z|^{l_2}}$$

*is a Toeplitz operator if and only if*

*(a) $l_2 - p \geq -1$, $l_1 - s \geq -1$ and $s = 0$ or 1;*

*or*

*(b) $\ell_1 = p = 0$ and/or $\ell_2 = s = 0$.*

**Proof:** First we apply Theorem 6.4 with $\varphi_1(z) = |z|^{\ell_1}$, and $\varphi_2(z) = |z|^{\ell_2}$ to see that the product $T_{e^{ip\theta}|z|^{l_1}}T_{e^{-is\theta}|z|^{l_2}}$ is a Toeplitz operator if and only if the function

$$\psi(z) = \begin{cases} \frac{l_2+s}{l_2+s-l_1-p}|z|^{l_2-p} - \frac{l_1+p}{l_2+s-l_1-p}|z|^{l_1-s} & \text{if } l_1 - s \neq l_2 - p \\ \\ |z|^{l_1-s}\big(1 + (l_1+p)\log|z|\big) & \text{if } l_1 - s = l_2 - p. \end{cases}$$

satisfies the conditions (i) and (ii) of the theorem. Looking at the definition of $\psi$, we see that $\psi$ is bounded or nearly bounded if the following condition is satisfied:

**(A):** ($\ell_2 + s \neq 0, \ell_1 + p \neq 0, \ell_2 - p \geq -1$ and $\ell_1 - s \geq -1$); or
($\ell_2 + s = 0$ and $\ell_1 - s \geq -1$); or
($\ell_1 + p = 0$ and $\ell_2 - p \geq -1$)

while $\psi$ is not even integrable if (A) is false. Thus we get that

$$(i) \iff (A).$$

Now, condition (ii) can be discussed only if (A) is true, otherwise the Mellin coefficients $\widehat{\psi}(m)(m \geq 2)$ are not all defined. But, in this case, a direct calculation shows that,

(I): If $\ell_2 + s \neq 0$ and $\ell_1 + p \neq 0$, then

$$\widehat{\psi}(m) = \frac{m-(s+p)}{(\ell_2 - p + m)(\ell_1 - s + m)} \qquad (m \geq 2).$$

(II):If $\ell_2 + s = 0$ and $\ell_1 + p \neq 0$, then

$$\widehat{\psi}(m) = \frac{\ell_1 + p}{(\ell_1 - s + m)(\ell_2 + s - \ell_1 - p)} \qquad (m \geq 2).$$

(III):If $\ell_2 + s \neq 0$ and $\ell_1 + p = 0$, then

$$\widehat{\psi}(m) = \frac{\ell_2 + s}{(\ell_2 - p + m)(\ell_2 + s - \ell_1 - p)} \qquad (m \geq 2).$$

(IV):If $\ell_2 + s = 0 = \ell_1 + p = 0$ then

$$\widehat{\psi}(m) = \frac{1}{m-(p+s)} \qquad (m \geq 2).$$

Thus, we see that

$$\widehat{\psi}(m) = 0 \text{ if and only if } m = p+s, \ell_2 + s \neq 0 \text{ and } \ell_1 + p \neq 0 \qquad (m \geq 2)$$

so that

$$\widehat{\psi}(2k+p-s+2) = 0 \text{ if and only if } k = s-1, \ell_2 + s \neq 0 \text{ and } \ell_1 + p \neq 0.$$

This shows that condition (ii) of Theorem 6.4 is verified if and only if:
"$s = 0$" (in which case condition (ii) is trivially satisfied)
or
"$s = 1$ and $\ell_2 + s \neq 0$ and $\ell_1 + p \neq 0$" (in which case $p - s + 2 = p + s$ so that condition (ii) requires only that $\widehat{\psi}(p+s) = 0$ which is true by (I)).

Thus we see that, if the product is a Toeplitz operator then, either

$$s = 0 \text{ and } (A)$$

or

$$s = 1, (A), \ell_2 + s \neq 0, \text{ and } \ell_1 + p \neq 0.$$

It is easy to see that these conditions imply that either (a) or (b) is true.

As for the sufficiency of conditions (a)and (b), if condition (a) is satisfied, then the product is a Toeplitz operator by Theorem 6.4 while, if condition (b) is satisfied then the product is also, clearly, a Toeplitz operator since, in this case, at least one of the two factors is the identity operator. This completes the proof.

Clearly an equivalent result can be obtained for $p < s$ by considering the adjoint of the operator and using Remark 6.2.

Corollary 6.5 corresponds to the result of Ahern discussed in Section 2, since, if $s, p \geq 0$, Corollary 6.5 (with $p = \ell_1$ and $s = \ell_2$) implies that, for $s \neq 0$ and

$p \neq 0$, $T_{z^p}T_{\overline{z}^s}$ is a Toeplitz operator if and only if $s = 1$ and $p = 1$, or 2. The same reasoning can be used to prove Ahern's theorem whenever $f_1$ and $g_2$ are polynomials.

We conclude with a theorem which illustrates the difficulty in characterizing more precisely those pairs of Toeplitz operators whose product is a Toeplitz operator. First, we prove a simple Lemma concerning Mellin transforms of polynomials.

**Lemma 6.6.** *For any $n$ and $t$ in $\mathbb{N}$ there exists a polynomial $q \neq 0$ such that:*

*(i)*

$$q(r) = r^n + a_1 r^{n+1} + \cdots + a_t r^{n+t};$$

*and*

*(ii)*

$$\hat{q}(2k+2) = 0 \qquad \textit{for } 0 \leq k \leq t-1.$$

**Proof:** By writing out the integrals defining the sequence $(\hat{q}(2k+2))_{k=0}^{t-1}$ for the polynomial of equation (i) we see that the existence of the polynomial $q$ is equivalent to the existence of a nonzero vector $v = (a_1, a_2, \cdots a_t)$ such that $Av = c$ where:

$$\mathbf{A} = \begin{bmatrix} \frac{1}{n+3} & \frac{1}{n+4} & \cdots & \frac{1}{n+t+2} \\ \frac{1}{n+5} & \frac{1}{n+6} & \cdots & \frac{1}{n+t+5} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n+2t+1} & \frac{1}{n+2t+2} & \cdots & \frac{1}{n+3t} \end{bmatrix}$$

and

$$\mathbf{c} = -\begin{pmatrix} \frac{1}{n+2} \\ \frac{1}{n+4} \\ \vdots \\ \frac{1}{n+2t} \end{pmatrix}.$$

Thus, what is required is the invertibility of the matrix $A = (a_{i,j})_{i,j=1}^t$ with

$$a_{i,j} = \frac{1}{n+2i+j}.$$

But this matrix is a 'Cauchy matrix' with determinant:

$$det(\mathbf{A}) = \frac{2^{\frac{s(s-1)}{2}}(1!2!\dots(s-2)!(s-1)!)^2}{\prod_{1\leq i,j\leq s}(n+2i+j)} \neq 0$$

(see [11], p. 36) and so the polynomial exists.

**Theorem 6.7.** *Let $p$ and $s$ be any two positive integers. Then:*

*(a) There exist $\phi_1$ and $\phi_2$ radial functions such that $e^{ip\theta}\phi_1$ and $e^{-is\theta}\phi_2$ are T-functions and*

$$T_{e^{ip\theta}\phi_1}T_{e^{-is\theta}\phi_2}$$

*IS a Toeplitz operator.*

*(b) There exist $\phi_1$ and $\phi_2$ radial functions such that $e^{ip\theta}\phi_1$ and $e^{-is\theta}\phi_2$ are T-functions and*

$$T_{e^{ip\theta}\phi_1}T_{e^{-is\theta}\phi_2}$$

*is not a Toeplitz operator.*

In the proof we shall refer to the 'minimal degree' of a analytic polynomial $p(r)$ which is defined (in the obvious way) to be the largest $n$ such that the quotient $\frac{p(r)}{r^n}$ is an analytic polynomial.

**Proof:** To prove (a) we first assume that $p \geq s$ and give an example of functions $\phi_1$ and $\phi_2$ such that $T_{e^{ip\theta}\phi_1}T_{e^{-is\theta}\phi_2}$ is a Toeplitz operator. The case $s > p$ then follows by considering the adjoint operator $T_{e^{is\theta}\overline{\phi_1}}T_{e^{-ip\theta}\overline{\phi_2}}$

Let $q$ be the polynomial of Lemma 6.6 with $n = p$ and $t = s$, let $\phi_1(z) = |z|^s$, and let $\phi_2(z) = |z|^s q(|z|)$. Then the function $\Lambda(r)$ of Theorem 6.4 will be

$$\Lambda(r) = r^{p+s} *_M r^{2s}q(r)$$

and, since $r^{2s}q(r)$ is a polynomial whose minimal degree is $p + 2s$, $\Lambda(r)$ is also a polynomial of minimal degree $p + s$. Thus, $\Lambda$ is differentiable and the function

$$\psi(t) = -t^{1-(p+s)}\Lambda'(t)$$

is also a polynomial. This means that $e^{i(p-s)\theta}\psi(z)$ is a T-function, and so part (i) of Theorem 6.4 is true.

As for (ii), we calculate directly the Mellin coefficients in question for $\psi$. We have:

$$\begin{aligned}\hat{\psi}(2k + p - s + 2) &= \int_0^1 -r^{1-(p+s)}\Lambda'(r)r^{2k+p-s+1}dr \\ &= -\int_0^1 \Lambda'(r)r^{2k-2s+2}dr \\ &= (2k - 2s + 2)\int_0^1 \Lambda(r)r^{2k-2s+1}dr\end{aligned}$$

using integration by parts and the fact that the function $\mu(t) = t^{2k+2-2s}\Lambda(t)$ satisfies $\mu(1) = \mu(0) = 0$. (This is where we use the assumption that $p \geq s$ which assures us that $\mu$ is a polynomial of of minimal degree $2k + 2 + p - s > 0$.) So, since

$$\Lambda(r) = r^{p+s} *_M r^{2s}q(r) = r^{2s}(r^{p-s} *_M q(r))$$

we see that:

$$\begin{aligned}\hat{\psi}(2k+p-s+2) =&(2k-2s+2)\int_0^1 (t^{p-s} *_M q)(r) r^{2k+1} dr\\ =&(2k-2s+2)\widehat{(r^{p-s} *_M q)}(2k+2)\\ =&(2k-2s+2)\widehat{r^{p-s}}(2k+2)\hat{q}(2k+2)=0\end{aligned}$$

for $k = 1, 2, ..., s-1$. Thus condition (ii) of Theorem 6.4 is also satisfied, and the product is, in fact, a Toeplitz operator.This proves (a).

There are of course lots of examples of functions $\phi_1$,$\phi_2$ such that

$$T_{e^{ip\theta}\phi_1} T_{e^{-is\theta}\phi_2}$$

is not a Toeplitz operator. If either $p$ or $s$ is greater than 1, one can take either $\ell_2 = p-2$ or $\ell_1 = s-2$ and Corollary 6.5 will show that, if $\phi_1(z) = |z|^{\ell_1}$ and $\phi_2(z) = |z|^{\ell_2}$ then the product $T_{e^{ip\theta}\phi_1} T_{e^{-is\theta}\phi_2}$ is not a Toeplitz operator. The cases $p = s = 1$ or $p = 1$ and $s = 0$ can be treated in the following way: We take $\phi_1(z) = \frac{1}{|z|}$ and $\phi_2(z) = \frac{1}{|z|}$. Then Theorem 6.4 shows that, if $T_{e^{ip\theta}\phi_1} T_{e^{-is\theta}\phi_2}$ were a Toeplitz operator, the symbol of this Toeplitz operator would be $e^{i(p-s)\theta}\frac{1}{|z|^2}$. But $e^{i(p-s)\theta}\frac{1}{|z|^2}$ is not a T-function, so the product is not equal to a Toeplitz operator.

Finally, suppose that $p = s = 0$. This is the most difficult case. The following construction was proposed by A. Borichev. The idea is that, if $h$ is a radial function in $L^1(\mathbb{D}, dA)$ then, for any $\gamma \in (0,1)$ the function $1 *_M h$, (considered as a function on $[0,1]$ is bounded on $[\gamma, 1]$ since for any $t$ in $[\gamma, 1]$ we have:

$$|(\mathbb{1} *_M h)(t)| \leq \int_t^1 |h(s)| \frac{ds}{s} \leq \frac{1}{\gamma^2} \|h\|_{L^1}. \tag{8}$$

So, if we find a T-function $f$ such that $f *_M f$ is not bounded on some interval $[\gamma, 1]$ then we will know that $f *_M f = \mathbb{1} *_M h$ has no solution in $L^1([0,1], rdr)$ which means that $T_f T_f$ is not a Toeplitz operator.

So, let $(t_k)_{k=0}^{\infty}$ be any sequence in $[\frac{1}{2}, 1)$ such that $t_k \longrightarrow 1$. Let $(\epsilon_k)_{k=0}^{\infty}$ be the sequence:

$$\epsilon_k = \min\big(t_k^{\frac{1}{2}} - t_k;\ (\frac{1}{2})^{3k}(1 - t_k^{\frac{1}{2}})^6\big).$$

and let $g$ be the $L^1([0,1], rdr)$ function defined by

$$g(s) = \sum_{k=0}^{\infty} \epsilon_k^{\frac{-2}{3}} \chi_{[t_k - \epsilon_k, t_k + \epsilon_k]}(s).$$

Then the mean value theorem gives us a real number $t'_k \in (t_k - \epsilon, t_k + \epsilon) \subseteq (t_k - \epsilon, t_k^{\frac{1}{2}})$ such that

$$n\hat{g}(n) = 2n \sum_{k=0}^{\infty} \epsilon_k^{\frac{1}{3}} (t'_k)^{n-1} \leq 2n \sum_{k=0}^{\infty} \epsilon_k^{\frac{1}{3}} (t_k^{\frac{1}{2}})^{n-1}.$$

Thus

$$\sum_{n=2}^{\infty} n\hat{g}(n) \leq 2\sum_{k=0}^{\infty}(\frac{1}{2})^k < \infty$$

and so $(n\hat{g}(n))_{n=2}^{\infty}$ is a bounded sequence. Considering $f$ to be the integrable radial function on $\mathbb{D}$ associated with $g$,this means that the (diagonal) operator $T_f$ is bounded so $f$ is a T-function and $T_f$ is a Toeplitz operator. But,

$$|(f *_M f)(t_k^2)| = \int_{t_k^2}^{1} g(s)g(\frac{t_k^2}{s})\frac{ds}{s} \geq \int_{t_k}^{t_k+\epsilon_k} g(s)g(\frac{t_k^2}{s})\frac{ds}{s} \geq \frac{\epsilon_k^{-\frac{1}{3}}}{\epsilon_k + t} \tag{9}$$

and the last term tends to $\infty$ as $k \to \infty$. Thus $T_fT_f$ is not a Toeplitz operator. This finishes the proof.

A rather different example of a radial T-function $f$ such that $T_fT_f$ is not equal to a Toeplitz operator can be found in [4] .

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