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On Toeplitz operators with quasihomogeneous symbols

By

ISSAM LOUHICHI and LOVA ZAKARIASY

Abstract. In this paper, we give some basic results concerning Toeplitz operators whose symbol is of the form $e^{ip\theta}\phi$, where ϕ is a radial function, then use these results to characterize all Toeplitz operators which commute with them.

1. Introduction. Let $\mathbb{D}$ denote the open unit disc in the complex plane and let $dA(z) = \frac{1}{\pi} r dr d\theta$ be normalized Lebesgue area measure on $\mathbb{D}$. The Bergman space L^2_a is the subset of $L^2(\mathbb{D}, dA)$, consisting of analytic functions on $\mathbb{D}$. Let P be the operator of orthogonal projection from $L^2(\mathbb{D}, dA)$ onto L^2_a. For a function $\varphi \in L^\infty(\mathbb{D}, dA)$, we define the Toeplitz operator $T_\varphi : L^2_a \mapsto L^2_a$ with symbol φ by

$$T_\varphi(f) = P(\varphi f).$$

It is well known that, if $K_z(w) = (1 - \bar{z}w)^{-2}$ is the Bergman reproducing kernel, then

$$T_\varphi(f)(z) = \int_{\mathbb{D}} \varphi(w) f(w) \overline{K_z(w)} dA(w), \quad z \in \mathbb{D}.$$

In 1964, Brown and Halmos [3] showed that on the Hardy space, two bounded Toeplitz operators T_φ and T_ψ commute if and only if: (i) both φ and ψ are analytic, or (ii) both $\bar{\varphi}$ and $\bar{\psi}$ are analytic, or (iii) one is a linear function of the other. In [1] Axler and Čučković proved that if the two symbols are bounded harmonic functions, then the same result is also true for Toeplitz operators on the Bergman space. Recently, with Rao [2], they proved that if φ is a bounded analytic function and if there exists a bounded function ψ such that T_φ and T_ψ commute on L^2_a, then ψ must be analytic too. In [8] and [9] Vasilevski gave the description of many (geometrically defined) classes of commuting Toeplitz operators.

The situation with a general symbol is rather more complicated. Let $\varphi, \psi \in L^1(\mathbb{D}, dA)$ be radial functions, i.e. $\varphi(z) = \varphi(|z|), z \in \mathbb{D}$. It is well known and easy to see that two

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Toeplitz operators with radial symbols commute. We show here that if p and s are integers such that $ps \leqq 0$, then the Toeplitz operators with symbols $e^{ip\theta}\varphi$ and $e^{is\theta}\psi$ commute only in certain trivial cases. The case $ps \geqq 0$ is treated for $\varphi = r^m$ in [4]. We note that the part (b) of Theorem 3 in [4] is not correct as stated.

2. Some basic results about quasihomogeneous symbols. An operator that will arise in our study of Toeplitz operators is the Mellin transform, defined for any function $\varphi \in L^1([0,1]; rdr)$, by the formula

$$\widehat{\varphi}(z) = \int_0^1 \varphi(r) r^{z-1} dr,$$

which is a bounded holomorphic function in the half plane $\{z : \operatorname{Re} z > 2\}$. It is known that if there exists a sequence $(n_k)_{k\geqq 0} \subset \mathbb{N}$, such that

$$\widehat{\varphi}(n_k) = 0 \text{ and } \sum_{k\geqq 0} \frac{1}{n_k} = \infty$$

then by the Muntz-Szasz theorem (see [7]), $\varphi = 0$.

A function f is said to be quasihomogeneous of degree k if and only if

$$f(re^{i\theta}) = e^{ik\theta}\varphi(r),$$

where φ is a radial function (see [6]). A direct calculation gives the following lemma which we shall use often.

Lemma 1. *Let $p \geqq 0$ an integer and φ a bounded radial function. Then, for all $n = 0, 1, 2, \ldots$:*

$$T_{e^{ip\theta}\varphi}(z^n) = 2(n+p+1)\widehat{\varphi}(2n+p+2)\, z^{n+p};$$

$$T_{e^{-ip\theta}\varphi}(z^n) = \begin{cases} 0 & \text{if } 0 \leqq n \leqq p-1 \\ 2(n-p+1)\widehat{\varphi}(2n-p+2)\, z^{n-p} & \text{if } n \geqq p. \end{cases}$$

Next, we introduce the notion of radialization (see [10]). If $f \in L^1(\mathbb{D}, dA)$, we define the "radialization" of f by

$$\operatorname{rad}(f)(z) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it}z)dt.$$

It is easy to see that a function f is radial if and only if $\operatorname{rad}(f) = f$.

Lemma 2. *For a bounded function f and for all integers $m,\ n \geqq 0$:*

$$\langle T_{\mathrm{rad}(f)} z^n, z^m \rangle = \begin{cases} \langle T_f z^m, z^m \rangle & \text{if } n = m \\ 0 & \text{if } n \neq m. \end{cases}$$

Proof. Writing out the integrals, changing the order of integration and making a change of variable, we have for all integers $m,\ n \geqq 0$,

$$\begin{aligned} \langle T_{\mathrm{rad}(f)} z^n, z^m \rangle &= \frac{1}{2\pi} \int_0^{2\pi} \int_{\mathbb{D}} f(e^{it} z) z^n \bar{z}^m \, dA(z)\, dt \\ &= \frac{1}{2\pi} \left(\int_0^{2\pi} e^{i(m-n)t} dt \right) \int_{\mathbb{D}} f(w) w^n \bar{w}^m \, dA(w) \\ &= \frac{1}{2\pi} \left(\int_0^{2\pi} e^{i(m-n)t} dt \right) \langle T_f z^n, z^m \rangle \\ &= \begin{cases} \langle T_f z^m, z^m \rangle & \text{if } n = m \\ 0 & \text{if } n \neq m. \end{cases} \quad \square \end{aligned}$$

Proposition 3. *A bounded function f is quasihomogeneous of degree $p \in \mathbb{Z}$ if and only if, for all integers $n \geqq 0$, there exists $\lambda_n \in \mathbb{C}$ such that*

$$T_f(z^n) = \begin{cases} 0 & \text{if } n < \max(-p, 0) \\ \lambda_n z^{n+p} & \text{if } n \geqq \max(-p, 0). \end{cases} \tag{1}$$

Proof. Let f be a quasihomogeneous function of degree p, i.e., $f = e^{ip\theta}\varphi$, where φ is a radial function. The necessity of condition (1) is a direct consequence of Lemma 1.

Conversely, suppose that equation (1) is true for any $p \in \mathbb{Z}$. Then, for any positive integers n and m:
if $p \geqq 0$

$$\langle T_{\bar{z}^p f} z^n, z^m \rangle = \langle f z^n, z^{m+p} \rangle = \begin{cases} 0 & \text{if } n \neq m \\ \dfrac{\lambda_m}{m+p+1} & \text{if } n = m; \end{cases}$$

and if $p < 0$

$$\langle T_{z^{-p} f} z^n, z^m \rangle = \langle f z^{n-p}, z^m \rangle = \begin{cases} 0 & \text{if } n \neq m \\ \dfrac{\lambda_{m-p}}{m+1} & \text{if } n = m. \end{cases}$$

Thus, if we note

$$\varphi = \begin{cases} \bar{z}^p f & \text{if } p \geqq 0 \\ z^{-p} f & \text{if } p < 0 \end{cases},$$

then by Lemma 2, we have for all integers $m, n \geqq 0$,

$$\langle T_{\mathrm{rad}(\varphi)} z^n, z^m \rangle = \langle T_\varphi z^n, z^m \rangle$$

hence, $T_{\mathrm{rad}(\varphi)} = T_\varphi$. Thus $\mathrm{rad}(\varphi) = \varphi$ and φ is a radial function. But this easily implies that f is a quasihomogeneous function of degree p. □

Proposition 4. *Let f_1 and f_2 be two quasihomogeneous bounded functions of degrees p and s respectively. If there exists a function h such that $T_{f_1} T_{f_2} = T_h$, then h is a quasihomogeneous function of degree $p + s$.*

Proof. Let φ_1 and φ_2 be the two radial functions, such that

$$f_1 = e^{ip\theta} \varphi_1 \quad \text{and} \quad f_2 = e^{is\theta} \varphi_2.$$

By Lemma 1, if p and s are greater than or equal to 0, then for all $n \geqq 0$, there exists $\lambda_n \in \mathbb{C}$ such that

$$T_{e^{ip\theta}\varphi_1} T_{e^{is\theta}\varphi_2}(z^n) = \lambda_n z^{n+p+s}.$$

When $p \geqq 0$ and $s < 0$, we have

$$T_{e^{ip\theta}\varphi_1} T_{e^{is\theta}\varphi_2}(z^n) = \begin{cases} 0 & \text{if } n < -s \\ \lambda_n z^{n+p+s} & \text{if } n \geqq -s. \end{cases}$$

So, if $T_{f_1} T_{f_2} = T_h$ then by Proposition 3, h is a quasihomogeneous function of degree $p + s$.

If p and s are both negative, or if $p < 0$ and $s \geqq 0$, then by considering the adjoint operator we obtain the same result. □

Proposition 4 helps us to identify the Toeplitz operators which are idempotents. The following corollary is an immediate consequence.

Corollary 5. *If f is a quasihomogeneous function of degree different from 0 and if $T_f^2 = T_f$, then $f = 0$.*

3. Commuting Toeplitz operators. In this section, we will see that two Toeplitz operators with quasihomogeneous symbols such that the signs of their quasihomogeneous degrees are opposite commute only in the trivial case, i.e., if one of them is the constant operator. Then Remark 12 shows that a function in $L^2(\mathbb{D}, dA)$ whose polar decomposition has only negative components cannot commute with a non trivial Toeplitz operator whose symbol is of positive quasihomogeneous degree.

Proposition 6. *Let ψ be a bounded radial function and $e^{ip\theta}\phi$ a quasihomogeneous bounded function of degree $p > 0$. If*

$$T_\psi T_{e^{ip\theta}\phi} = T_{e^{ip\theta}\phi} T_\psi$$

then, $\phi = 0$ or ψ is a constant.

Proof. If $T_{e^{ip\theta}\phi}$ and T_ψ commute, then for all $n \geqq 0$,

$$\begin{aligned}&(2n+2p+2)\widehat{\phi}(2n+p+2)\widehat{\psi}(2n+2p+2)\\ &\quad = (2n+2)\widehat{\phi}(2n+p+2)\widehat{\psi}(2n+2).\end{aligned}$$

Let $E = \{n : \widehat{\phi}(2n+p+2) = 0\}$. If $\sum_{n\in E} \frac{1}{n} = \infty$, then $\phi = 0$, as discussed in Section 2. Otherwise, $\sum_{n\in E^c} \frac{1}{n} = \infty$, where E^c is the complement of E in $\mathbb{N}$, and so we have

$$(2n+2p+2)\widehat{\psi}(2n+2p+2) = (2n+2)\widehat{\psi}(2n+2),\ \forall n \in E^c.$$

This implies that

$$(z+2p)\widehat{\psi}(z+2p) = z\widehat{\psi}(z), \forall z \in \{\mathrm{Re}\, z > 0\}.$$

For any integer n_0 greater than 0, the last equation gives us

$$(n_0+2kp)\widehat{\psi}(n_0+2kp) = n_0\widehat{\psi}(n_0), \quad \forall k \in \mathbb{N}.$$

If we denote by C the constant $n_0\widehat{\psi}(n_0)$, we obtain

$$\widehat{\psi}(n_0+2kp) = \frac{C}{n_0+2kp} = C\widehat{\mathbb{1}}(n_0+2kp), \quad \forall k \in \mathbb{N}$$

and so, ψ is equal to $C\mathbb{1}$. □

Remark 7. If $p < 0$ the same result is true by considering the adjoint of the operator.

Proposition 8. *Let $p \geqq s$ be two integers greater than 0 and ϕ and ψ two bounded radial functions. If*

$$T_{e^{ip\theta}\phi} T_{e^{-is\theta}\psi} = T_{e^{-is\theta}\psi} T_{e^{ip\theta}\phi}$$

then, $\phi = 0$ or $\psi = 0$.

Proof. If $T_{e^{ip\theta}\phi}$ and $T_{e^{-is\theta}\psi}$ commute, then, for each $n \geqq 0$,

(a) $\widehat{\phi}(2n+p+2)\widehat{\psi}(2n+2p-s+2) = 0$, if $n \leqq s-1$;
(b) $\widehat{\phi}(2n+p+2)\widehat{\psi}(2n+2p-s+2) = C_n\widehat{\phi}(2n+p-2s+2)\widehat{\psi}(2n-s+2)$ if $n \geqq s$.

where $C_n = \frac{n-s+1}{n+p+1}$.

Now, equation (a) implies that for all $n_0 \leqq s-1$, there exists a sequence $(n_k)_{k\in\mathbb{N}}$, which is defined by $n_{k+1} = n_k + s$ or $n_k + p$, such that

$$\widehat{\phi}(2n_k + p + 2)\widehat{\psi}(2n_k + 2p - s + 2) = 0. \tag{2}$$

It is clear that

$$\sum_{k\in\mathbb{N}} \frac{1}{2n_k+1} = \infty.$$

Let $E_1 = \{k : \widehat{\phi}(2n_k + p + 2) = 0\}$ and $E_2 = \{k : \widehat{\psi}(2n_k + 2p - s + 2) = 0\}$. Since

$$\sum_{k\in\mathbb{N}} \frac{1}{2n_k+1} \leqq \sum_{k\in E_1} \frac{1}{2n_k+1} + \sum_{k\in E_2} \frac{1}{2n_k+1},$$

we see that at least one of the series $\sum\limits_{k\in E_1} \frac{1}{2n_k+1}$ and $\sum\limits_{k\in E_2} \frac{1}{2n_k+1}$ diverges, and so $\phi = 0$ or $\psi = 0$. □

This result is not true if both of the integers p and s are positive. There are lots of examples of functions of positive quasihomogeneous degree which are the symbols of commuting Toeplitz operators (see [4]). In the general case we have the following proposition.

Proposition 9. *Let p, s be two integers greater than 0 and $\phi \neq 0$ a bounded radial function. If there exists a bounded radial function ψ not identically zero, such that*

$$T_{e^{ip\theta}\phi} T_{e^{is\theta}\psi} = T_{e^{is\theta}\psi} T_{e^{ip\theta}\phi}$$

then, ψ is unique up to a constant factor.

Proof. Suppose there exist two functions $\psi_1 \neq 0$ and $\psi_2 \neq 0$, such that

$$\begin{aligned} T_{e^{ip\theta}\phi} T_{e^{is\theta}\psi_1} &= T_{e^{is\theta}\psi_1} T_{e^{ip\theta}\phi}, \\ T_{e^{ip\theta}\phi} T_{e^{is\theta}\psi_2} &= T_{e^{is\theta}\psi_2} T_{e^{ip\theta}\phi}. \end{aligned}$$

This is equivalent to:

$$\begin{cases} \widehat{\phi}(2n + p + 2s + 2)\widehat{\psi_1}(2n + s + 2) \\ \quad = C_n \widehat{\phi}(2n + p + 2)\widehat{\psi_1}(2n + 2p + s + 2) \\ \widehat{\phi}(2n + p + 2s + 2)\widehat{\psi_2}(2n + s + 2) \\ \quad = C_n \widehat{\phi}(2n + p + 2)\widehat{\psi_2}(2n + 2p + s + 2) \end{cases}$$

with $C_n = \frac{n+p+1}{n+s+1}$. From these equalities, one obtains:

$$\widehat{r^s\psi_1}(2n + 2)\widehat{r^s\psi_2}(2n + 2p + 2) = \widehat{r^s\psi_1}(2n + 2p + 2)\widehat{r^s\psi_2}(2n + 2) \tag{3}$$

for all n in the set $E = \{n \in \mathbb{N} : \widehat{\phi}(2n + 2s + p + 2)\widehat{\phi}(2n + p + 2) \neq 0\}$.

Clearly, we have $\sum_{n\in E}\frac{1}{n}=\infty$ because we supposed that $\phi\neq 0$.

From this and equation (3), we have

$$\widehat{r^s\psi_1}(z)\widehat{r^s\psi_2}(z+2p)=\widehat{r^s\psi_1}(z+2p)\widehat{r^s\psi_2}(z),\ \forall z\in\{\mathrm{Re}\, z>-s\}. \tag{4}$$

Now, since ψ_1 is not identically zero, there exists an integer n_0 such that

$$\widehat{r^s\psi_1}(n_0)\neq 0.$$

Let $F=\{k\in\mathbb{N}:\widehat{r^s\psi_1}(n_0+2kp)=0\}$. If $\sum_{k\in F}\frac{1}{n_0+2kp}=\infty$, then equation (4) implies that $\widehat{r^s\psi_2}(n_0+2kp)=0$ for all $k\in F$. Thus, $r^s\psi_2=0$ and so $\psi_2=0$. Otherwise, we have $\sum_{k\in F^c}\frac{1}{n_0+2kp}=\infty$ and so by equation (4) we obtain that

$$(\widehat{r^s\psi_2}-\lambda\widehat{r^s\psi_1})(n_0+2kp)=0,\quad \text{where } \lambda=\frac{\widehat{r^s\psi_2}(n_0)}{\widehat{r^s\psi_1}(n_0)}.$$

Thus, in this case, $\psi_2=\lambda\psi_1$. □

Now, let $\mathcal{R}$ be the space of functions which are square integrable in $[0,1]$ with respect to the measure rdr. By using the fact that the trigonometric polynomials are dense in $L^2(\mathbb{D},dA)$ and that for $k_1\neq k_2$, $e^{ik_1\theta}\mathcal{R}$ is orthogonal to $e^{ik_2\theta}\mathcal{R}$, one see that:

$$L^2(\mathbb{D},dA)=\bigoplus_{k\in\mathbb{Z}}e^{ik\theta}\mathcal{R}.$$

Thus, each function $\psi\in L^2(\mathbb{D},dA)$ can be written as (see [4]):

$$\psi(re^{i\theta})=\sum_{k\in\mathbb{Z}}e^{ik\theta}\psi_k(r),\quad \text{where } \psi_k\in\mathcal{R}. \tag{5}$$

Moreover, if $\psi\in L^\infty(\mathbb{D},dA)\subset L^2(\mathbb{D},dA)$ then for each $r\in[0,1)$,

$$|\psi_k(r)|=\left|\int_0^{2\pi}\psi(re^{i\theta})e^{-ik\theta}\frac{d\theta}{2\pi}\right|\leqq\sup_{z\in\mathbb{D}}|\psi(z)|,\quad \forall k\in\mathbb{Z}$$

hence, the functions ψ_k are bounded in the disk.

Lemma 10. *Let $e^{ip\theta}\phi$ be a bounded function of quasihomogeneous degree $p\geqq 0$ and let*

$$\psi(re^{i\theta})=\sum_{k\in\mathbb{Z}}e^{ik\theta}\psi_k(r)\in L^\infty(\mathbb{D},dA).$$

Then,

$$T_\psi T_{e^{ip\theta}\phi}=T_{e^{ip\theta}\phi}T_\psi\iff T_{e^{ik\theta}\psi_k}T_{e^{ip\theta}\phi}=T_{e^{ip\theta}\phi}T_{e^{ik\theta}\psi_k},\quad \forall k\in\mathbb{Z}.$$

P r o o f. Since

$$T_\psi T_{e^{ip\theta}\phi}(z^n) = \sum_{k+p+n\geqq 0} T_{e^{ik\theta}\psi_k} T_{e^{ip\theta}\phi} z^n$$

and

$$T_{e^{ip\theta}\phi} T_\psi (z^n) = \sum_{k+n\geqq 0} T_{e^{ip\theta}\phi} T_{e^{ik\theta}\psi_k} z^n$$

then, for each couple $(m, n) \in \mathbb{N}^2$:

$$\langle T_\psi T_{e^{ip\theta}\phi} z^n, z^m\rangle = \langle T_{e^{i(m-n-p)\theta}\psi_{m-n-p}} T_{e^{ip\theta}\phi} z^n, z^m\rangle \tag{6}$$

and

$$\langle T_{e^{ip\theta}\phi} T_\psi z^n, z^m\rangle = \langle T_{e^{ip\theta}\phi} T_{e^{i(m-n-p)\theta}\psi_{m-n-p}} z^n, z^m\rangle. \tag{7}$$

On the other hand,

$$\begin{aligned}
&\langle T_{e^{ip\theta}\phi} T_{e^{ik\theta}\psi_k} z^n, z^m\rangle \\
&\quad = \begin{cases} 0 & \text{if } k \neq m-n-p \\ \langle T_{e^{ip\theta}\phi} T_{e^{i(m-n-p)\theta}\psi_{m-n-p}} z^n, z^m\rangle & \text{if } k = m-n-p \end{cases} \\
&\quad = \begin{cases} 0 & \text{if } k \neq m-n-p \\ \langle T_{e^{ip\theta}\phi} T_\psi z^n, z^m\rangle & \text{if } k = m-n-p \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
&\langle T_{e^{ik\theta}\psi_k} T_{e^{ip\theta}\phi} z^n, z^m\rangle \\
&\quad = \begin{cases} 0 & \text{if } k \neq m-n-p \\ \langle T_{e^{i(m-n-p)\theta}\psi_{m-n-p}} T_{e^{ip\theta}\phi} z^n, z^m\rangle & \text{if } k = m-n-p \end{cases} \\
&\quad = \begin{cases} 0 & \text{if } k \neq m-n-p \\ \langle T_\psi T_{e^{ip\theta}\phi} z^n, z^m\rangle & \text{if } k = m-n-p \end{cases}.
\end{aligned}$$

This shows that, if $T_{e^{ip\theta}\phi} T_\psi = T_\psi T_{e^{ip\theta}\phi}$, then for each $(m, n) \in \mathbb{N}^2$ and for all $k \in \mathbb{Z}$,

$$\langle T_{e^{ip\theta}\phi} T_{e^{ik\theta}\psi_k} z^n, z^m\rangle = \langle T_{e^{ik\theta}\psi_k} T_{e^{ip\theta}\phi} z^n, z^m\rangle,$$

and so $T_{e^{ip\theta}\phi} T_{e^{ik\theta}\psi_k} = T_{e^{ik\theta}\psi_k} T_{e^{ip\theta}\phi},\ \forall k \in \mathbb{Z}$.

Conversely, if for all $k \in \mathbb{Z}$, $T_{e^{ip\theta}\phi}T_{e^{ik\theta}\psi_k} = T_{e^{ik\theta}\psi_k}T_{e^{ip\theta}\phi}$, then in particular, for any pair $m,\ n \in \mathbb{N}$, if $k = m - n - p$, we have:

$$\langle T_{e^{ip\theta}\phi}T_{e^{i(m-n-p)\theta}\psi_{m-n-p}}z^n, z^m\rangle = \langle T_{e^{i(m-n-p)\theta}\psi_{m-n-p}}T_{e^{ip\theta}\phi}z^n, z^m\rangle.$$

Thus, by formulas (6) and (7),

$$\langle T_{e^{ip\theta}\phi}T_\psi z^n, z^m\rangle = \langle T_\psi T_{e^{ip\theta}\phi}z^n, z^m\rangle$$

for all m and n in $\mathbb{N}$. Finally, $T_{e^{ip\theta}\phi}T_\psi = T_\psi T_{e^{ip\theta}\phi}$. □

Theorem 11. *Let $e^{ip\theta}\phi$ be a non-zero bounded function of quasihomogeneous degree $p > 0$. If*

$$\psi(re^{i\theta}) = \sum_{k\in\mathbb{Z}} e^{ik\theta}\psi_k(r) \in L^\infty(\mathbb{D}, dA)$$

is such that $T_\psi T_{e^{ip\theta}\phi} = T_{e^{ip\theta}\phi}T_\psi$, then $\psi_k = 0$ for all $k < 0$.

Proof. By Lemma 10, $T_\psi T_{e^{ip\theta}\phi} = T_{e^{ip\theta}\phi}T_\psi$ implies that $T_{e^{ik\theta}\psi_k}$ commutes with $T_{e^{ip\theta}\phi}$ for each $k \in \mathbb{Z}$. Thus, if $k < 0$ then the Proposition 8 gives $\psi_k = 0$ for all $k < 0$. □

Remark 12. Let ψ be as in Theorem 11 and ϕ a non constant bounded radial function such that $T_\psi T_\phi = T_\phi T_\psi$. Then, Lemma 10 gives us $T_\phi T_{e^{ik\theta}\psi_k} = T_{e^{ik\theta}\psi_k}T_\phi$ for all $k \in \mathbb{Z}$. Now, Proposition 6 implies that $\psi_k = 0$ for all $k > 0$, while Remark 7 shows that $\psi_k = 0$ for all $k < 0$. So $\psi = \psi_0$.

This remark gives another proof of Theorem 6 in [4].

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I. Louhichi
UFR de Mathématiques Informatiques
Université Bordeaux I
351 cours de la libération
F-33405 Talence Cedex
France
louhichi@math.u-bordeaux1.fr

L. Zakariasy
High Institute of Technology
201 Diego-Suarez
Madagascar
zakarias@refer.mg