ROOTS OF TOEPLITZ OPERATORS ON THE BERGMAN SPACE

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ABSTRACT. One of the major questions in the theory of Toeplitz operators on the Bergman space over the unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$ is a complete description of the commutant of a given Toeplitz operator, that is the set of all Toeplitz operators that commute with it. In [4], the first author obtained a complete description of the commutant of Toeplitz operator T with any quasihomogeneous symbol $\phi(r)e^{ip\theta}, p > 0$ in case it has a Toeplitz p-th root S with symbol $\psi(r)e^{i\theta}$, namely, commutant of T is the closure of the linear space generated by powers S^n which are Toeplitz. But the existence of p-th root was known until now only when $\phi(r) = r^m, m \geq 0$. In this paper we will show the existence of p-th roots for a much larger class of symbols, for example, it includes such symbols for which

$$\phi(r) = \sum_{i=1}^{k} r^{a_i}(\ln r)^{b_i}, 0 \leq a_i, b_i \text{ for all } 1 \leq i \leq k.$$

1. INTRODUCTION

Let $\mathbb{D}$ be the unit disc in the complex plane $\mathbb{C}$, and $dA = rdr\frac{d\theta}{\pi}$ be the normalized Lebesgue area measure so that the measure of $\mathbb{D}$ equals 1. Let L_a^2 be The Bergman space, the Hilbert space of functions, analytic on $\mathbb{D}$ and square integrable with respect to the measure dA. We denote the inner product in $L^2(\mathbb{D}, dA)$ by $<,>$. It is well known that L_a^2 is a closed subspace of the Hilbert space $L^2(\mathbb{D}, dA)$, with the set of functions $\{\sqrt{n+1}z^n \mid n \geq 0\}$ as an orthonormal basis. Let P be the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto L_a^2. For a bounded function f on $\mathbb{D}$, the Toeplitz operator T_f with symbol f is defined by

$$T_f(h) = P(fh) \text{ for } h \in L_a^2.$$

A symbol f is said to be quasihomogeneous of order p an integer, if it can be written as $f(re^{i\theta}) = e^{ip\theta}\phi(r)$, where ϕ is a radial function on $\mathbb{D}$. In this case, the associated Toeplitz operator T_f is also called quasihomogeneous Toeplitz of order p. Quasihomogeneous Toeplitz operators were first introduced in [2] while generalizing the results of [1]. We assume $p > 0$ from now on.

We are looking for, given a quasihomogeneous operator T of degree p, a quasihomogeneous operator S of degree 1 such that $S^p = T$. It was proved in [4] that any such root if it exists, is unique up to a multiplicative constant. Also the existence of $p-$th roots for the case $\phi(r) = r^m$ for any arbitrary $m \geq 0, p > 0$ was proved in [4] using the results in [1]. Here we plan to deal with more general $\phi(r)$.

Date: July 3, 2010.

2. The Mellin Transform and two Lemmas

For any two functions $f(r)$ and $g(r)$ defined on $I = [0,1]$, we define the Mellin convolution as follows:

$$(f * g)(r) = \int_r^1 f(\frac{r}{t})g(t)\frac{dt}{t}.$$

Often we are interested in knowing when the convolution is a bounded function in the interval I. To that purpose, we introduce the following concept of the type for a function f. We say f is of type (a, b) with $a \geq 0$ and $b > 0$ if

$$|f(r)| \leq Cr^a(1-r)^{b-1}$$

on I, where C is a constant depending on f. Also we express the same thing as

$$f(r) \ll r^a(1-r)^{b-1}$$

omitting the constants and the absolute value signs.

Lemma A. Suppose $f(r)$ is of type (a, b) and $g(r)$ is of type (c, d). Then their convolution product

$$\begin{cases} (f * g) \text{ is of type } (\min\{a,c\}, b+d) & \text{if } a \neq c \\ \text{and} & \\ (f * g)(r) \ll r^{\min\{a,c\}}(1-r)^{b+d-1}\ln(\frac{e}{r}) & \text{if } a = c. \end{cases}$$

This can be generalized to any finite product as follows: Suppose for $1 \leq i \leq n$, $f_i(r)$ is of type (a_i, b_i). Then $h(r)$, their convolution product satisfies

$$h(r) \ll r^\alpha(1-r)^{\beta-1}\left(\ln\left(\frac{e}{r}\right)\right)^{n-1} \tag{1}$$

where $\alpha = \min\{a_i\}, \beta = \sum b_i$. Further, if we know the number of a_i that are equal to $\min\{a_i\}$ to be say l, the estimate (1) can be improved to

$$h(r) \ll r^\alpha(1-r)^{\beta-1}\left(\ln\left(\frac{e}{r}\right)\right)^{l-1}. \tag{2}$$

Thus the log term will disappear if $l = 1$.

Remark 1. Most of the time our aim is to prove h is bounded and the presence of log does not interfere with that aim since $\alpha > 0$ which makes $h(r)$ bounded near zero and since $\beta \geq 1$, it is bounded near 1. But log cannot be avoided. Take for example $f_i(r) = r$ for every i and compute the convolution product. It checks out to be $\frac{r(\ln r)^{n-1}}{(n-1)!}$, by a simple integration.

Lemma B. Suppose $f_i(r) = r^{a_i}(1-r)^{b_i-1}$ where a_i, b_i are positive for $1 \leq i \leq n$. Let α, β be as defined in Lemma A. Given any integer $k \geq 0$, the $k-$the derivative of h, the convolution product of f_i, satisfies the following:

$$h^{(k)}(r) \ll r^{\alpha-k}(1-r)^{\beta-k-1}\left(\ln\left(\frac{e}{r}\right)\right)^{n-1}.$$

Here the constant involved depends on k and h.

3. Applications of Lemmas A and B

One of our most useful tools in the following calculations will be the Mellin transform. The Mellin transform $\widehat{\phi}$ of a radial function ϕ in $L^1([0,1], rdr)$ is defined by

$$\widehat{\phi}(z) = \int_0^1 f(r) r^{z-1}\, dr = \mathcal{M}(\phi)(z).$$

It is well known that, for these functions, the Mellin transform is well defined on the right half-plane $\{z : \Re z \geq 2\}$ and it is analytic on $\{z : \Re z > 2\}$. It is important and helpful to know that the Mellin transform $\widehat{\phi}$ is uniquely determined by its values on any arithmetic sequence of integers. In fact we have the following classical theorem [8, p.102].

Theorem 1. *Suppose that f is a bounded analytic function on $\{z : \Re z > 0\}$ which vanishes at the pairwise distinct points $z_1, z_2 \cdots$, where*

i) $\inf\{|z_n|\} > 0$
   and
ii) $\sum_{n\geq 1} \Re(\frac{1}{z_n}) = \infty$.

Then f vanishes identically on $\{z : \Re z > 0\}$.

Remark 2. Now one can apply this theorem to prove that if $\phi \in L^1([0,1], rdr)$ and if there exist $n_0, p \in \mathbb{N}$ such that

$$\widehat{\phi}(pk + n_0) = 0 \text{ for all } k \in \mathbb{N},$$

then $\widehat{\phi}(z) = 0$ for all $z \in \{z : \Re z > 2\}$ and so $\phi = 0$.

Moreover, it is easy to see that the Mellin transform converts the convolution product into a pointwise product, i.e that:

$$\widehat{(\phi * \psi)}(r) = \widehat{\phi}(r)\widehat{\psi}(r).$$

A direct calculation shows that a quasihomogeneous Toeplitz operator acts on the elements of the orthogonal basis of L^2_a as a shift operator with a holomorphic weight. In fact, for $p \geq 0$ and for all $k \geq 0$, we have

$$\begin{aligned} T_{e^{ip\theta}\phi}(z^k) &= P(e^{ip\theta}\phi z^k) = \sum_{n\geq 0}(n+1)\langle e^{ip\theta}\phi z^k, z^n\rangle z^n \\ &= \sum_{n\geq 0}(n+1)\int_0^1\int_0^{2\pi} \phi(r) r^{k+n+1} e^{i(k+p-n)\theta}\frac{d\theta}{\pi} dr z^n \\ &= 2(k+p+1)\widehat{\phi}(2k+p+2) z^{k+p}. \end{aligned}$$

Now we are ready to start with the following relatively easy example.

3.1. **p-th roots of $T_{e^{ip\theta}\phi}$ where $\phi(r) = r + r^2$.** Does there exist a radial function ψ such that $\left(T_{e^{i\theta}\psi}\right)^p = T_{e^{ip\theta}\phi}$? If it is the case, then we will have

$$\left(T_{e^{i\theta}\psi}\right)^p (z^k) = T_{e^{ip\theta}\phi}(z^k), \text{ for all } k \geq 0.$$

Since

$$\left(T_{e^{i\theta}\psi}\right)^p (z^k) = \left[\prod_{j=0}^{p-1}(2k+2j+4)\widehat{\psi}(2k+2j+3)\right] z^{k+p},$$

we obtain for all integers $k \geq 0$

$$(2k+2p+2)\widehat{\phi}(2k+p+2) = \left[\prod_{j=0}^{p-1}(2k+2j+4)\widehat{\psi}(2k+2j+3)\right],$$

from which and Remark 2 follows,, by setting $z = 2k+3$, the identity, valid in the right halfplane

$$(z+2p-1)\widehat{\phi}(z+p-1) = \left[\prod_{j=0}^{p-1}(z+2j+1)\widehat{\psi}(z+2j)\right]. \tag{1}$$

If we divide the equation (1) by the equation obtained by replacing z by $z+2$ in the equation (1), after cancelation, we obtain that in the right halfplane,

$$\frac{\widehat{\psi}(z+2p)}{\widehat{\psi}(z)} = \frac{(z+1)\widehat{\phi}(z+p+1)}{(z+2p-1)\widehat{\phi}(z+p-1)}, \text{ for } \Re z > 0. \tag{2}$$

Since $\widehat{\phi}(z) = \dfrac{1}{z+1} + \dfrac{1}{z+2} = \dfrac{2z+3}{(z+1)(z+2)}$, it follows that

$$\frac{\widehat{\psi}(z+2p)}{\widehat{\psi}(z)} = \frac{(z+1)}{(z+2p-1)}\frac{(2z+2p+5)}{(z+p+2)(z+p+3)}\frac{(z+p)(z+p+1)}{(2z+2p+1)}, \text{ for } \Re z > 0.$$

If we denote by $\lambda(\zeta) = \widehat{\psi}(2p\zeta)$, the above equation becomes

$$\frac{\lambda(\zeta+1)}{\lambda(\zeta)} = \frac{(2p\zeta+1)(4p\zeta+2p+5)(2p\zeta+p)(2p\zeta+p+1)}{(2p\zeta+2p-1)(2p\zeta+p+2)(2p\zeta+p+3)(4p\zeta+2p+1)}, \text{ for } \Re\zeta > 0.$$

Using the well-known identity $\Gamma(z+1) = z\Gamma(z)$, where Γ is the Gamma function, we can write that

$$\frac{\lambda(\zeta+1)}{\lambda(\zeta)} = \frac{F(\zeta+1)}{F(\zeta)} \text{ for } \Re\zeta > 0, \tag{3}$$

where

$$F(\zeta) = \frac{\Gamma(\zeta+a_1)\Gamma(\zeta+a_2)\Gamma(\zeta+a_3)\Gamma(\zeta+a_4)}{\Gamma(\zeta+a_1')\Gamma(\zeta+a_2')\Gamma(\zeta+a_3')\Gamma(\zeta+a_4')},$$

with a_i are in increasing order $\dfrac{2}{4p}, \dfrac{2p}{4p}, \dfrac{2p+2}{4p}, \dfrac{2p+5}{4p}$ respectively and a_i' are in almost increasing order $\dfrac{2p+1}{4p}, \dfrac{2p+4}{4p}, \dfrac{4p-2}{4p}, \dfrac{2p+6}{4p}$ respectively for $i = 1, \ldots, 4$. Equation (3), combined with [4, Lemma 6, p.1428], gives us that there exists a constant C such that

$$\lambda(\zeta) = CF(\zeta), \text{ for } \Re\zeta > 0. \tag{4}$$

Basic observation is that the quotient of two Gamma functions

$$\frac{\Gamma(\zeta+a_i)}{\Gamma(\zeta+a_i')}, \text{ where } 0 < a_i < a_i'$$

is a constant times the Beta function

$$B(\zeta+a_i, a_i'-a_i) = \int_0^1 x^{\zeta+a_i-1}(1-x)^{a_i'-a_i-1}\,dx.$$

Moreover, according to our definition of the Mellin transform, it turns out that $B(\zeta + a_i, a'_i - a_i)$ is the Mellin Transform of $x^{a_i}(1-x)^{a'_i - a_i - 1}$ which is of type $(a_i, a'_i - a_i)$. Since the a_i are smaller than a'_i respectively for $i = 1, \ldots, 4$ (in fact $a'_3 \geq a_3$ if and only if $2p \geq 4$ which is always true), Equation (4) implies that

$$\lambda(\zeta) = C\Pi_{i=1}^{4} B(\zeta + a_i, a'_i - a_i),$$

where C is a constant. Since the product of Mellin transforms equals to the Mellin of the convolution product, we would have

$$\lambda(\zeta) = Ch(\zeta),$$

where h is the convolution product of four functions of type $(a_i, a'_i - a_i)$, $i = 1, \ldots, 4$. Now Lemma A tells us that

$$h(r) \ll r^{\min\{a_i\}}(1-r)^{\sum_i (a'_i - a_i) - 1} \ln(\frac{e}{r}).$$

Because $\sum_i a'_i - a_i = 1$, we have

$$h(r) \ll r^{\min\{a_i\}} \ln(\frac{e}{r}),$$

and hence h is bounded function. Therefore the function ψ, if it exists, satisfies the equation

$$\widehat{\psi}(2p\zeta) = C\widehat{h}(\zeta)$$

for some constant C, which is equivalent to

$$\int_0^1 \psi(r) r^{2p\zeta - 1} dr = C \int_0^1 h(t) t^{\zeta - 1} dt.$$

Now, by a change of variables $t = r^{2p}$, we obtain

$$\int_0^1 \psi(r) r^{2p\zeta - 1} dr = \int_0^1 h(r^{2p}) r^{2p\zeta - 1} 2p dr.$$

Thus $\psi(r) = 2ph(r^{2p})$, and so ψ is bounded. Hence the operator $T_{e^{i\theta}\psi}$ is a genuine Toeplitz operator and $p-$th root of $T_{e^{ip\theta}\phi}$.

3.2. **p-th roots of $T_{e^{ip\theta}\phi}$ where $\widehat{\phi}(z)$ is a proper rational fraction.** We recall that if there exists a radial function ψ such that $\left(T_{e^{i\theta}\psi}\right)^p = T_{e^{ip\theta}\phi}$, then we have Equation (2) which is

$$\widehat{\psi}(z + 2p) = \widehat{\psi}(z) \frac{(z+1)\widehat{\phi}(z+p+1)}{(z+2p-1)\widehat{\phi}(z+p-1)}, \text{ for } \Re z > 0.$$

Here we are assuming $\widehat{\phi}(z) = \frac{P(z)}{Q(z)}$ where $P(z) = \prod_{j=1}^{m}(z + a_j)$ and $Q(z) = \prod_{k=1}^{n}(z + b_k)$ with $1 \leq m < n$. So that

$$\begin{aligned} \widehat{\psi}(z + 2p) &= \widehat{\psi}(z) \frac{(z+1)}{(z+2p-1)} \frac{P(z+p+1)Q(z+p-1)}{P(z+p-1)Q(z+p+1)} \\ &= \frac{(z+1)}{(z+2p-1)} \prod_{j=1}^{m} \frac{z + a_j + p + 1}{z + a_j + p - 1} \prod_{k=1}^{n} \frac{z + b_k + p - 1}{z + b_k + p + 1} \end{aligned}$$

Let $\lambda(\zeta) = \widehat{\psi}(2p\zeta)$. Then the equality above becomes

$$\frac{\lambda(\zeta+1)}{\lambda(\zeta)} = \frac{(2p\zeta+1)}{(2p\zeta+2p-1)} \prod_{j=1}^{m} \frac{2p\zeta + a_j + p + 1}{2p\zeta + a_j + p - 1} \prod_{k=1}^{n} \frac{2p\zeta + b_k + p - 1}{2p\zeta + b_k + p + 1}$$

Therefore, by [4, Lemma 6, p.1428], λ is constant times the quotient of $m+n+1$ Gamma functions in the numerator and about the same in the denominator as follows:

$$\lambda(\zeta) = C\frac{\Gamma(\zeta + A_0)}{\Gamma(\zeta + A_0')} \prod_{j=1}^{m} \frac{\Gamma(\zeta + A_j)}{\Gamma(\zeta + A_j')} \prod_{k=1}^{n} \frac{\Gamma(\zeta + B_k)}{\Gamma(\zeta + B_k')} \tag{5}$$

where $A_0 = \frac{1}{2p}$, $A_0' = \frac{2p-1}{2p}$, $A_j = \frac{a_j+p+1}{2p}$, $A_j' = \frac{a_j+p-1}{2p}$, $B_k = \frac{b_k+p-1}{2p}$ and $B_k' = \frac{b_k+p+1}{2p}$ for $1 \le j \le m$ and $1 \le k \le n$. Based on the same argument as in the previous subsection, we would like to write each quotient of two Gamma functions as a constant times a Beta function. In order to do that, we must assume that all A_j and B_k are positive for every $0 \le j \le m$ and $1 \le k \le n$. Moreover we observe that

$$A_0' - A_0 = \frac{p-1}{p},\ A_j' - A_j = -\frac{1}{p},\ B_k' - B_k = \frac{1}{p}.$$

So each quotient of two Gamma functions in Equation (5)can be written as a constant times a Beta function except those involving A_j for $1 \le j \le m$. We fix this matter by noting that $\Gamma(\zeta + A_j' + 1) = (\zeta + A_j')\Gamma(\zeta + A_j')$, and so here $A_j' + 1 - A_j = \frac{p-1}{p}$. Hence, Equation (5) becomes

$$\frac{\lambda(\zeta)}{\prod_{j=1}^{m}(\zeta + A_j')} = C\frac{\Gamma(\zeta + A_0)}{\Gamma(\zeta + A_0')} \prod_{j=1}^{m} \frac{\Gamma(\zeta + A_j)}{\Gamma(\zeta + A_j' + 1)} \prod_{j=1}^{n} \frac{\Gamma(\zeta + B_j)}{\Gamma(\zeta + B_j')}.$$

As in the previous subsection, this quotient of $m+n+1$ Gamma functions on the numerator and the same in the denominator, respectively would be the Mellin transform of the convolution product of $m+n+1$ functions. Let us denoted it h. By Lemma A, we have

$$h(r) \ll r^A (1-r)^{B-1} \left(\ln\left(\frac{e}{r}\right)\right)^{m+n},$$

where $A = \min\{A_j\}$ which is definitely positive, and B is given by

$$A_0' - A_0 + \sum_{j=1}^{m} A_j' + 1 - A_j + \sum_{k=1}^{n} B_k' - B_k = (m+1)\frac{p-1}{p} + \frac{n}{p} = m + 1 + \frac{n-m-1}{p}.$$

Therefore we obtain

$$h(r) \ll r^A (1-r)^{m + \frac{n-m-1}{p}} \left(\ln\left(\frac{e}{r}\right)\right)^{m+n} = r^A (1-r)^{m+v} \left(\ln\left(\frac{e}{r}\right)\right)^{m+n},$$

where $v = \dfrac{n-m-1}{p}$ is a non-negative number. Using Lemma B, we see that h has all derivatives of order not exceeding m and they satisfy the inequality

$$r^j h^{(j)}(r) \ll r^A (1-r)^{m-j+v} \left(\ln\left(\frac{e}{r}\right)\right)^{m+n}.$$

Further the function ψ, if it exists, would satisfy the equation

$$\widehat{\psi}(2p\zeta) = C\left(\prod_{j=1}^{m}(\zeta + A'_j)\right)\widehat{h}(\zeta). \tag{6}$$

Now it is easy to check by integration by parts the following identity

$$\zeta\widehat{h}(\zeta) = -\mathcal{M}\left(r\frac{dh}{dr}\right)(\zeta)$$

provided h vanishes at 1 and rh' is bounded in $(0,1)$. Thus in the current case, denoting h' by Dh where $D = \dfrac{d}{dr}$, we can see

$$(\zeta + A'_j)\widehat{h}(\zeta) = \mathcal{M}\left(\left(A'_j - rD\right)h\right)(\zeta),$$

and

$$\left(\prod_{j=1}^{m}(\zeta + A'_j)\right)\widehat{h}(\zeta) = \mathcal{M}\left(\prod_{j=1}^{m}(A'_j - rD)h\right)(\zeta).$$

Let us set

$$H(r) = \left(\prod_{j=1}^{m}(A'_j - rD)h\right)(r)$$

which allows us to rewrite Equation (6) as

$$\int_0^1 \psi(r)r^{2p\zeta-1}dr = C\int_0^1 H(t)t^{\zeta-1}\,dt.$$

Now, by a change of variables $t = r^{2p}$, we obtain

$$\int_0^1 \psi(r)r^{2p\zeta-1}dr = C\int_0^1 H(r^{2p})r^{2p\zeta-1}2pdr.$$

Thus $\psi(r) = 2pCH(r^{2p})$, and hence is bounded and the operator $T_{e^{i\theta}\psi}$ is a genuine Toeplitz operator and $p-$th root of $T_{e^{ip\theta}\phi}$.

4. Proof of the Lemma A for two functions

We choose to start proving Lemma A for two functions f and g of type (a,b) and (c,d) respectively, with a, b, c and d are all positive. Similar thing was discussed in [1, pages 210-212] but with less generality since the goal was different.

Let $h(r) = (f * g)(r)$. By definition of the Mellin convolution, it is easy to see that

$$h(r) \ll \int_r^1 \left(\frac{r}{t}\right)^a\left(1-\frac{r}{t}\right)^{b-1} t^c(1-t)^{d-1}\,\frac{dt}{t},$$

which after a change of variables $\dfrac{t-r}{1-r} = u$ and using the consequent identities

$$t = r + u - ru =,\ t - r = u(1-r),\ 1-t = (1-u)(1-r),\ dt = (1-r)du$$

leads to

$$\begin{aligned}
h(r) &\ll \int_r^1 \left(\frac{r}{t}\right)^a \left(1-\frac{r}{t}\right)^{b-1} t^c(1-t)^{d-1}\,\frac{dt}{t}\\
&= \int_r^1 \left(\frac{r}{t}\right)^a \left(\frac{t-r}{t}\right)^{b-1} t^c(1-t)^{d-1}\,\frac{dt}{t}\\
&= \int_0^1 r^a t^{-a} u^{b-1}(1-r)^{b-1} t^{-b+1} t^c (1-u)^{d-1}(1-r)^{d-1}(1-r)\frac{du}{t}\\
&= r^a(1-r)^{b+d-1}\int_0^1 t^{c-a-b}u^{b-1}(1-u)^{d-1}du.
\end{aligned}$$

We have the following cases

- If $c-a-b\geq 0$. Since $0\leq t\leq 1$, we have
$$h(r)\ll r^a(1-r)^{b+d-1},$$
and hence h is of type $(a,b+d)$.
- If $c-a-b<0$. Assuming $c-a>0$ and noting that $t\geq u$, we obtain
$$\begin{aligned}
h(r) &\ll r^a(1-r)^{b+d-1}\int_0^1 u^{c-a-b}u^{b-1}(1-u)^{d-1}\,du\\
&\leq r^a(1-r)^{b+d-1}\int_0^1 u^{c-a-1}(1-u)^{d-1}\,du\\
&= r^a(1-r)^{b+d-1}B(c-a,d),
\end{aligned}$$
and therefore h is of type $(a,b+d)$.

Now in case $c=a$, for any number $0<\epsilon\leq b$, noticing that $t\geq r$ and $u>0$, we have

$$\begin{aligned}
h(r) &= r^a(1-r)^{b+d-1}\int_0^1 t^{-b}u^{b-1}(1-u)^{d-1}\,du\\
&\ll r^a(1-r)^{b+d-1}\int_0^1 t^{-\epsilon}t^{\epsilon-b}u^{b-1}(1-u)^{d-1}\,du\\
&\leq r^a(1-r)^{b+d-1}\int_0^1 r^{-\epsilon}u^{\epsilon-b}u^{b-1}(1-u)^{d-1}\,du\\
&\leq r^a(1-r)^{b+d-1}r^{-\epsilon}\int_0^1 u^{\epsilon-1}(1-u)^{d-1}\,du\\
&\leq r^a(1-r)^{b+d-1}B(\epsilon,d)r^{-\epsilon}.
\end{aligned}$$

Now since $\epsilon B(\epsilon,d)=\frac{\Gamma(\epsilon+1)\Gamma(d)}{\Gamma(\epsilon+d)}$ is holomorphic as a function of ϵ in a neighborhood of the interval $(0,b)$, there exists a constant C such that $\epsilon B(\epsilon,d)\leq C$ on that interval, and therefore

$$h(r)\leq Cr^a(1-r)^{b+d-1}r^{-\epsilon}\epsilon^{-1},\ \text{for every } 0<\epsilon\leq b.$$

Here we emphasize the fact that C does not depend on r and ϵ as long as $0<r<1$ and $0<\epsilon\leq b$. For a fixed but arbitrary r, let $E(\epsilon)=r^{-\epsilon}\epsilon^{-1}$ and $m(r)=\min\limits_{(0,b]}E(\epsilon)$. Then

$$h(r)\leq Cr^a(1-r)^{b+d-1}m(r). \tag{7}$$

Moreover the function E decreases in the interval $\left(0, -\frac{1}{\ln r}\right)$ and increases in the interval $\left(-\frac{1}{\ln r}, +\infty\right)$. Further $-\frac{1}{\ln r} \le b$ if and only if $r \le e^{-\frac{1}{b}}$. Thus Equation (7) implies

If $r \le e^{-\frac{1}{b}}$,

$$h(r) \ll r^a(1-r)^{b+d-1}m(r) \le r^a(1-r)^{b+d-1}e\ln\left(\frac{1}{r}\right). \tag{8}$$

If $r > e^{-\frac{1}{b}}$,

$$h(r) \ll r^a(1-r)^{b+d-1}r^{-b}b^{-1} \le r^a(1-r)^{b+d-1}\frac{e}{b}. \tag{9}$$

Combining (8) and (9), we obtain

$$\begin{aligned} h(r) &\ll r^a(1-r)^{b+d-1}\left(e\ln\left(\frac{1}{r}\right)+\frac{e}{b}\right) \\ &\ll r^a(1-r)^{b+d-1}\ln\left(\frac{e}{r}\right), \text{ for all } 0<r<1. \end{aligned}$$

5. Lemma A for convolution product of more than two functions

In this context we can assume that the function f_i which is of type (a_i, b_i) is

$$f_i(x) = x^{a_i}(1-x)^{b_i-1}, \text{ for } 1 \le i \le n.$$

The convolution product of these n functions is defined by a repeated integral

$$\begin{aligned} h(r) &= \int_r^1 \int_{r/x_1}^1 \int_{r/x_1x_2}^1 \cdots \int_{r/x_1x_2\ldots x_{n-2}}^1 \\ &f_1(x_1)f_2(x_2)\ldots f_{n-1}(x_{n-1})f_n\left(\frac{r}{x_1\ldots x_{n-1}}\right)\frac{dx_{n-1}}{x_{n-1}}\cdots\frac{dx_3}{x_3}\frac{dx_2}{x_2}\frac{dx_1}{x_1} \end{aligned} \tag{10}$$

As in the case of two functions where we made a change of variables $u = \frac{t-r}{1-r}$, we make change of variables so that the new integral is over the unit cube I^{n-1} where limits of integration do not depend on other variables. Let $y_0 = 1$ and inductively define $y_i = \prod_{j=1}^{i} x_i$ for $i \ge 1$. Now we make the change of variables as follows:

$$x_i = \frac{r}{y_{i-1}} + \left(1 - \frac{r}{y_{i-1}}\right)\xi_i, \text{ for } i \ge 1,$$

so that the limits for each ξ_i are 0 and 1. Further we note

$$y_i - r = x_i y_{i-1} - r = (y_{i-1} - r)\xi_i, \text{ for } i \ge 0.$$

Let us set $\eta_0 = 1$ and $\eta_i = \prod_{j=1}^{i}\xi_i$, for $i \ge 1$. It is easy to show, by induction on i, that

$$y_i - r = (1-r)\eta_i, \text{ for all } i \ge 1.$$

Further

$$(1-x_i) = (1-\xi_i)\left(1-\frac{r}{y_{i-1}}\right) = \frac{(1-\xi_i)(1-r)\eta_{i-1}}{y_{i-1}}, \text{ for all } i \ge 1. \tag{11}$$

Thus

$$f_i(x_i) = x_i^{a_i}(1-x_i)^{b_i-1} = \left(\frac{y_i}{y_{i-1}}\right)^{a_i}\left(\frac{(1-\xi_i)(1-r)\eta_{i-1}}{y_{i-1}}\right)^{b_i-1}, \text{ for } 1 \le i \le n-1.$$

But for $i = n$, we have

$$\begin{aligned} f_n(x_n) &= \left(\frac{r}{y_{n-1}}\right)^{a_n}\left(1-\frac{r}{y_{n-1}}\right)^{b_n-1} \\ &= \left(\frac{r}{y_{n-1}}\right)^{a_n}\left(\frac{y_{n-1}-r}{y_{n-1}}\right)^{b_n-1} \\ &= \left(\frac{r}{y_{n-1}}\right)^{a_n}\left(\frac{(1-r)\eta_{n-1}}{y_{n-1}}\right)^{b_n-1}. \end{aligned}$$

Writing the product of functions in (10) in terms of ξ_i, η_i, r, y_i, for $1 \le i \le n-1$ yields

$$r^{a_n}\prod_{i=1}^{n-1}\eta_i^{b_{i+1}-1}\prod_{i=1}^{n-1}y_i^{a_i-a_{i+1}-b_{i+1}+1}\prod_{i=1}^{n-1}(1-\xi_i)^{b_i-1}\prod_{i=1}^{n}(1-r)^{b_i-1}. \tag{12}$$

Using equalities of (11), we calculate the differential form

$$\bigwedge_{i=1}^{n-1}\frac{dx_i}{x_i} = \bigwedge_{i=1}^{n-1}\frac{(1-r)\eta_{i-1}d\xi_i}{x_i y_{i-1}} = (1-r)^{n-1}\prod_{i=1}^{n-1}\frac{\eta_{i-1}}{y_i}\bigwedge_{i=1}^{n-1}d\xi_i. \tag{13}$$

From (10), (11), and (12) we derive

$$\begin{aligned} h(r) &= r^{a_n}(1-r)^{b_1+\ldots+b_n-1}\int_{I^{n-1}}\eta_{n-1}^{b_n-1}\prod_{i=1}^{n-2}\eta_i^{b_{i+1}} \\ &\qquad \prod_{i=1}^{n-1}y_i^{a_i-a_{i+1}-b_{i+1}}\prod_{i=1}^{n-1}(1-\xi_i)^{b_i-1}\bigwedge_{i=1}^{n-1}d\xi_i. \end{aligned} \tag{14}$$

Let us assume that a_i are arranged in decreasing order. Then the product

$$\eta_i^{b_{i+1}}y_i^{a_i-a_{i+1}-b_{i+1}} \le 1$$

since $\eta_i \le y_i \le 1$. Therefore

$$h(r) \le r^{a_n}(1-r)^{b_1+\ldots+b_n-1}\int_{I^{n-1}}\eta_{n-1}^{b_n-1}y_{n-1}^{a_{n-1}-a_n-b_n}\prod_{i=1}^{n-1}(1-\xi_i)^{b_i-1}\bigwedge_{i=1}^{n-1}d\xi_i.$$

Here four cases have to be discussed.

Case 1. If $a_{n-1}-a_n-b_n \ge 0$. Then

$$\begin{aligned} h(r) &\le r^{a_n}(1-r)^{b_1+\ldots+b_n-1}\int_{I^{n-1}}\eta_{n-1}^{b_n-1}\prod_{i=1}^{n-1}(1-\xi_i)^{b_i-1}\bigwedge_{i=1}^{n-1}d\xi_i \\ &\le r^{a_n}(1-r)^{b_1+\ldots+b_n-1}\prod_{i=1}^{n-1}\frac{\Gamma(b_n)\Gamma(b_i)}{\Gamma(b_n+b_i)}. \end{aligned}$$

Case 2. If $a_{n-1}-a_n-b_n<0$ and $a_{n-1}\neq a_n$. Then

$$\begin{aligned} h(r) &\leq r^{a_n}(1-r)^{b_1+\ldots+b_n-1}\int_{I^{n-1}} y_{n-1}^{a_{n-1}-a_n-b_n}\eta_{n-1}^{b_n-1}\prod_{i=1}^{n-1}(1-\xi_i)^{b_i-1}\bigwedge_{i=1}^{n-1}d\xi_i \\ &\leq r^{a_n}(1-r)^{b_1+\ldots+b_n-1}\int_{I^{n-1}} \eta_{n-1}^{a_{n-1}-a_n-b_n}\eta_{n-1}^{b_n-1}\prod_{i=1}^{n-1}(1-\xi_i)^{b_i-1}\bigwedge_{i=1}^{n-1}d\xi_i \\ &\leq r^{a_n}(1-r)^{b_1+\ldots+b_n-1}\int_{I^{n-1}} \eta_{n-1}^{a_{n-1}-a_n-1}\prod_{i=1}^{n-1}(1-\xi_i)^{b_i-1}\bigwedge_{i=1}^{n-1}d\xi_i \\ &\leq r^{a_n}(1-r)^{b_1+\ldots+b_n-1}\prod_{i=1}^{n-1}\frac{\Gamma(a_{n-1}-a_n)\Gamma(b_i)}{\Gamma(a_{n-1}-a_n+b_i)}. \end{aligned}$$

Case 3. If $a_{n-1}=a_n$. Choose an arbitrary $0<\epsilon\leq b_n$ and note that $y_{n-1}\geq r$, and $\eta_{n-1}>0$. Therefore

$$\begin{aligned} h(r) &\leq r^{a_n}(1-r)^{b_1+\ldots+b_n-1}\int_{I^{n-1}} y_{n-1}^{-b_n}\eta_{n-1}^{b_n-1}\prod_{i=1}^{n-1}(1-\xi_i)^{b_i-1}\bigwedge_{i=1}^{n-1}d\xi_i \\ &\leq r^{a_n}(1-r)^{b_1+\ldots+b_n-1}\int_{I^{n-1}} y_{n-1}^{-\epsilon}y_{n-1}^{\epsilon-b_n}\eta_{n-1}^{b_n-1}\prod_{i=1}^{n-1}(1-\xi_i)^{b_i-1}\bigwedge_{i=1}^{n-1}d\xi_i \\ &\leq r^{a_n}(1-r)^{b_1+\ldots+b_n-1}\int_{I^{n-1}} r^{-\epsilon}\eta_{n-1}^{\epsilon-b_n}\eta_{n-1}^{b_n-1}\prod_{i=1}^{n-1}(1-\xi_i)^{b_i-1}\bigwedge_{i=1}^{n-1}d\xi_i \\ &\leq r^{a_n}(1-r)^{b_1+\ldots+b_n-1}\int_{I^{n-1}} r^{-\epsilon}\eta_{n-1}^{\epsilon-1}\prod_{i=1}^{n-1}(1-\xi_i)^{b_i-1}\bigwedge_{i=1}^{n-1}d\xi_i \\ &= r^{a_n}(1-r)^{b_1+\ldots+b_n-1}r^{-\epsilon}\prod_{i=1}^{n-1}\frac{\Gamma(\epsilon)\Gamma(b_i)}{\Gamma(\epsilon+b_i)}. \end{aligned}$$

As we did in section 2, this product of quotients of Gamma functions is meromorphic on the interval $[0,b_n]$ except at zero where it has a pole of order $n-1$, and so there exists a constant C such that

$$\prod_{i=1}^{n-1}\frac{\Gamma(\epsilon)\Gamma(b_i)}{\Gamma(\epsilon+b_i)}\leq C\epsilon^{1-n}.$$

Hence

$$h(r)\ll r^{a_n}(1-r)^{b_1+\ldots+b_n-1}r^{-\epsilon}\epsilon^{1-n}.$$

Now

- If $r\leq e^{-\frac{1}{b_n}}$, then $\dfrac{1}{\ln\left(\frac{1}{r}\right)}\leq b_n$. In this case, we choose $\epsilon=\dfrac{1}{\ln(\frac{1}{r})}$ and obtain

$$h(r)\ll r^{a_n}(1-r)^{b_1+\ldots+b_n-1}e\left(\ln\left(\frac{1}{r}\right)\right)^{n-1}. \tag{15}$$

- If $r\geq e^{-\frac{1}{b_n}}$, we choose $\epsilon=b_n$ and obtain

$$\begin{aligned} h(r) &\ll r^{a_n}(1-r)^{b_1+\ldots+b_n-1}r^{-b_n}b_n^{1-n} \\ &\leq r^{a_n}(1-r)^{b_1+\ldots+b_n-1}eb_n^{1-n}. \end{aligned} \tag{16}$$

Combining (15) and (16) together yields

$$\begin{aligned} h(r) &\ll r^{a_n}(1-r)^{b_1+\ldots+b_n-1}\left(e\left(\ln\left(\frac{1}{r}\right)\right)^{n-1}+eb_n^{1-n}\right)\\ &\ll r^{a_n}(1-r)^{b_1+\ldots+b_n-1}\left(\ln\left(\frac{1}{r}\right)\right)^{n-1}, \text{ for all } 0<r<1. \end{aligned}$$

Case 4. Assume there exists k such that $a_k > a_{k+1} = \ldots = a_n = a$. Let $F(r)$ be the convolution product of $f_1, f_2, \ldots, f_{k+1}$ and $G(r)$ be the convolution product of the rest namely $f_{k+2}\ldots, f_n$. From the previous discussion it is clear that

$$F(r) \ll r^a(1-r)^{b_1+\ldots+b_{k+1}-1}$$

and

$$G(r) \ll r^a(1-r)^{b_{k+2}+\ldots+b_n-1}\left(\ln\left(\frac{e}{r}\right)\right)^{n-k-2}.$$

Denote by $b = b_1+\ldots+b_{k+1}$, $d = b_{k+2}+\ldots+b_n$ and $n-k-1 = l$. The case $l=1$ has been treated previously. So assume $l>1$. We see that

$$\begin{aligned} h(r) &= (F*G)(r)\\ &\ll \int_r^1 (r/t)^a(1-r/t)^{b-1}t^a(1-t)^{d-1}\left(\ln\left(\frac{e}{t}\right)\right)^{l-1}\frac{dt}{t}\\ &\le r^a\int_r^1 (t-r)^{b-1}t^{-b}(1-t)^{d-1}\left(\ln\left(\frac{e}{t}\right)\right)^{l-1}\frac{dt}{t}. \end{aligned}$$

Now the change of variables $t = u+r-ur$ leads to $t-r = u(1-r)$, $1-t = (1-u)(1-r)$, $dt = (1-r)du$ and

$$\begin{aligned} h(r) &\ll r^a\int_0^1 (1-r)^{b-1}u^{b-1}t^{-b}(1-r)^{d-1}(1-u)^{d-1}\left(\ln\left(\frac{e}{t}\right)\right)^{l-1}(1-r)\,du\\ &\le r^a(1-r)^{b+d-1}\int_0^1 u^{b-1}t^{-b}(1-u)^{d-1}\left(\ln\left(\frac{e}{t}\right)\right)^{l-1}du. \end{aligned}$$

Noting that $t\ge u$ and $r>0$ and choosing an arbitrary $0<\epsilon\le b$ implies

$$\begin{aligned} h(r) &\ll r^a(1-r)^{b+d-1}\int_0^1 u^{b-1}t^{-\epsilon}t^{\epsilon-b}(1-u)^{d-1}\left(\ln\left(\frac{e}{t}\right)\right)^{l-1}du\\ &\le r^a(1-r)^{b+d-1}\int_0^1 u^{b-1}r^{-\epsilon}u^{\epsilon-b}(1-u)^{d-1}\left(\ln\left(\frac{e}{t}\right)\right)^{l-1}du\\ &\le r^a(1-r)^{b+d-1}r^{-\epsilon}\int_0^1 u^{b-1}u^{\epsilon-b}(1-u)^{d-1}\left(\ln\left(\frac{e}{u}\right)\right)^{l-1}du\\ &\le r^a(1-r)^{b+d-1}r^{-\epsilon}\int_0^1 u^{\epsilon-1}(1-u)^{d-1}\left(\ln\left(\frac{e}{u}\right)\right)^{l-1}du. \end{aligned}$$

Let $H_j(\epsilon) = \displaystyle\int_0^1 u^{\epsilon-1}(1-u)^{d-1}(\ln u)^j du$. This is the $j-$th order derivative of the beta function $B(\epsilon, d)$ as a function of ϵ, and $B(\epsilon, d)$ is holomorphic on $(-1,\infty)$ except at zero where it has a simple pole with residue 1. This is easy to verify. So $\epsilon^{j+1} H_j(\epsilon)$ will be holomorphic on the interval $(-1, \infty)$. Observing that

$$\int_0^1 u^{\epsilon-1}(1-u)^{d-1}\left(\ln\left(\frac{e}{u}\right)\right)^{l-1} du$$

is a linear sum of the derivatives of order less than or equal to $l-1$ of the Beta function, we find

$$\epsilon^l \int_0^1 u^{\epsilon-1}(1-u)^{d-1}\left(\ln\left(\frac{e}{u}\right)\right)^{l-1} du$$

is bounded by a constant C in the interval $[0, b]$. Thus

$$h(r) \ll r^a(1-r)^{b+d-1} r^{-\epsilon}\epsilon^{-l}.$$

Now arguing as in Case 3, if $r \le e^{-\frac{1}{b}}$, we choose $\epsilon = \frac{1}{\ln\left(\frac{1}{r}\right)}$ and get

$$h(r) \ll r^a(1-r)^{b+d-1} e\left(\ln\left(\frac{1}{r}\right)\right)^l$$

and if $r > e^{-\frac{1}{b}}$, we let $\epsilon = b$, and have

$$h(r) \ll r^a(1-r)^{b+d-1}\frac{e}{b^l}.$$

Combining these two cases, we obtain

$$h(r) \ll r^a(1-r)^{b+d-1}\left(\ln\left(\frac{e}{r}\right)\right)^l.$$

This totally proves the Lemma A.

6. Proof of Lemma B

We recall (14)

$$\begin{aligned} h(r) &= r^{a_n}(1-r)^{b_1+\ldots+b_n-1}\int_{I^{n-1}} \eta_{n-1}^{b_n-1}\prod_{i=1}^{n-2}\eta_i^{b_{i+1}} \\ &\quad \prod_{i=1}^{n-1} y_i^{a_i-a_{i+1}-b_{i+1}}\prod_{i=1}^{n-1}(1-\xi_i)^{b_i-1}\bigwedge_{i=1}^{n-1} d\xi_i. \end{aligned}$$

To make the differentiation easier, we introduce some notation. Let

$A = a_n,\ B = b_1+\ldots+b_n,\ \eta = (\eta_1, \ldots, \eta_{n-1}),\ \xi = (\xi_1, \ldots, \xi_{n-1}),\ y = (y_1, \ldots, y_{n-1})$

$\alpha_i = a_i - a_{i+1} - b_{i+1}$ for $1 \le i \le n-1$, $\beta_i = b_{i+1}$ for $1 \le i \le n-2$, $\beta_{n-1} = b_n - 1$,

$$\beta = (\beta_1, \ldots, \beta_{n-1}),\ G(\xi) = \prod_{i=1}^{n-1}(1-\xi_i)^{b_i-1},\ d\xi = \bigwedge_{i=1}^{n-1} d\xi_i,\ J = I^{n-1}.$$

With this notation and the multi-index notation like for example $y^\alpha = y_1^{\alpha_1}\ldots y_{n-1}^{\alpha_{n-1}}$, (14) can be written as

$$h(r) = r^A(1-r)^{B-1}\int_J y^\alpha \eta^\beta G(\xi)\, d\xi. \tag{17}$$

Clearly the function $\eta^\beta G(\xi_i)$ is summable $d\xi$, and each $y_i = \eta_i + r(1-\eta_i)$ satisfies $0 < r \le y_i < 1$ for $0 < r < 1$. So one can differentiate under the integral sign with respect to r. But before we do that let us introduce some more notation

$$g_1(r) = r^A,\ g_2(r) = (1-r)^{B-1},\ u_i = y_i^{\alpha_i} \text{ for } 1 \le i \le n-1.$$

Rewriting (17) as

$$h(r) = \int_J g_1 g_2 u_1 \dots u_{n-1} \eta^\beta G(\xi)\, d\xi. \tag{18}$$

Now differentiating under the integral sign, we obtain

$$h^{(k)}(r) = \sum \int_J g_1^{(l_1)} g_2^{(l_2)} u_1^{(j_1)} \dots u_{n-1}^{(j_{n-1})} \eta^\beta G(\xi)\, d\xi \tag{19}$$

where the summation is taken over all $(n+1)-$tuples of non-negative integers $(l_1, l_2, j_1, \dots j_{n-1})$ such that $k = l_1 + l_2 + j_1 + \dots + j_{n-1}$. Further it easy to check the following

$$\begin{aligned}
u_i^{(j_i)}(r) &= \alpha_i(\alpha_i - 1)\dots(\alpha_i - j_i + 1) y_i^{\alpha_i - j_i}(1-\eta_i)^{j_i},\\
g_1^{(l_1)}(r) &= A(A-1)\dots(A - l_1 + 1) r^{A - l_1},\\
g_2^{(l_2)}(r) &= (B-1)(B-2)\dots(B-l_2)(-1)^{l_2}(1-r)^{B-l_2-1}.
\end{aligned}$$

Since $y_i \ge r$ and $0 \le \eta_i \le 1$, the equalities above imply

$$u_i^{(j_i)}(r) \ll y_i^{\alpha_i} r^{-j_i} \tag{20}$$

$$g_1^{(l_1)}(r) \ll g_1(r) r^{-l_1} \tag{21}$$

$$g_2^{(l_2)}(r) \ll (1-r)^{B-k-1} = g_2(r)(1-r)^{-k} \tag{22}$$

where the last inequality is obtained because $0 \le l_2 \le k$. From (20), (21) and (22) we deduce that

$$\begin{aligned}
g_2^{(l_2)}(r) g_1^{(l_1)}(r) u_1^{(j_1)}(r) \dots u_{n-1}^{(j_{n-1})}(r) &\ll g_2(r)(1-r)^{-k} g_1(r) u_1(r) \dots\\
&\qquad \dots u_{n-1}(r) r^{-l_1 - j_1 - \dots - j_{n-1}}\\
&\ll r^{-k}(1-r)^{-k} g_2(r) g_1(r) u_1(r) \dots u_{n-1}(r).
\end{aligned}$$

Multiplying both sides by $\eta^\beta G(\xi) d\xi$ and integrating over J yield

$$h^{(k)}(r) \ll r^{-k}(1-r)^{-k} h(r)$$

and by the Lemma A,

$$h(r) \ll r^A (1-r)^{B-1} \left(\ln\left(\frac{e}{r}\right)\right)^{n-1}.$$

Hence we have

$$h^{(k)}(r) \ll r^{A-k}(1-r)^{B-k-1} \left(\ln\left(\frac{e}{r}\right)\right)^{n-1}.$$

This proves the Lemma B.

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