ON SOME PROBLEMS FOR TOEPLITZ AND TRUNCATED TOEPLITZ OPERATORS

MÜBARIZ T. KARAEV AND ISSAM LOUHICHI

Abstract. For a scalar inner function θ, the model space of Sz.-Nagy and Foias is the subspace $K_\theta = H^2 \ominus \theta H^2$ of the classical Hardy space $H^2 = H^2(\mathbb{D})$ over the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. For a bounded linear operator A on the model space K_θ, its Berezin symbol is the function $\widetilde{A}^{K_\theta}$ defined on $\mathbb{D}$ by $\widetilde{A}^{K_\theta}(\lambda) = \left\langle A\widehat{k}_{\theta,\lambda}, \widehat{k}_{\theta,\lambda} \right\rangle$, where

$$\hat{k}_{\theta,\lambda}(z) = \left(\frac{1-|\lambda|^2}{1-|\theta(\lambda)|^2}\right)^{1/2} \frac{1-\overline{\theta(\lambda)}\theta(z)}{1-\overline{\lambda}z}$$

is the normalized reproducing kernel of the subspace K_θ. We shall consider the following question: Let $A : K_\theta \to K_\theta$ is an operator for which there exists a constant $\delta > 0$ such that $\left|\widetilde{A}^{K_\theta}(\lambda)\right| \geq \delta > 0$,for all $\lambda \in \mathbb{D}$. Under which additional conditions is A invertible ? In this article we investigate this question in the case where θ is an interpolation Blaschke product. In particular, the invertibility property of truncated Toeplitz operators is investigated. We also give further related results on the Toeplitz operators on the Bergman space $L_a^2(\mathbb{D})$.

1. Introduction

In this paper we continue the investigation of a generalized Douglas problem started by the first author in [20]. We consider the question of invertibility of operators on the model space $K_\theta = H^2 \ominus \theta H^2$ of Sz.-Nagy and Foias, where $H^2 = H^2(\mathbb{D})$ is the Hardy space of all analytic functions in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ for which

$$\|f\|_2^2 \overset{def}{=} \sup_{0 \leq r < 1} \int_{\mathbb{T}} |f(r\zeta)|^2 \, dm(\zeta) < \infty,$$

where $\mathbb{T} = \partial\mathbb{D} = \{\zeta : |\zeta| = 1\}$ is the unit circle, m is the normalized Lebesgue measure on $\mathbb{T}$, and θ is an inner function (i.e., $\theta \in H^2$ and $|\theta(\zeta)| = 1$ a.a. $\zeta \in \mathbb{T}$). In particular, the invertibility of the truncated Toeplitz operators $\mathcal{T}_{\varphi,\theta} := P_\theta T_\varphi | K_\theta$, $\varphi \in L^\infty(\mathbb{T})$, is investigated. For more information about the theory of truncated Toeplitz operators, see for example [3] and [25, 26].

2000 *Mathematics Subject Classification.* Primary 47A35; Secondary 47B20.
Key words and phrases. Model space, Shvedenko constant, Interpolation Blaschke product, Reproducing kernel, Berezin symbol, Toeplitz operator.

Recall that H^2 is a reproducing kernel Hilbert space, with the kernel

$$k_\lambda(z) = \frac{1}{1-\overline{\lambda}z} \ \ (\lambda, z \in \mathbb{D})$$

known as the Szegö kernel. Thus $\langle f, k_\lambda \rangle = f(\lambda)$ for all $f \in H^2$ and $\lambda \in \mathbb{D}$. Therefore the function

$$k_{\theta,\lambda}(z) \overset{def}{=} P_\theta k_\lambda(z) = \frac{1-\overline{\theta(\lambda)}\theta(z)}{1-\overline{\lambda}z} \ \ (\lambda, z \in \mathbb{D})$$

where P_θ is the orthogonal projection from H^2 onto K_θ, is the reproducing kernel for the space K_θ. For any bounded linear operator $A : K_\theta \to K_\theta$, the Berezin symbol of A is the function $\widetilde{A}^{K_\theta}(\lambda)$ on $\mathbb{D}$ defined by the formula

$$\widetilde{A}^{K_\theta}(\lambda) \overset{def}{=} \left\langle A\widehat{k}_{\theta,\lambda}, \widehat{k}_{\theta,\lambda} \right\rangle, \ \lambda \in \mathbb{D},$$

where

$$\widehat{k}_{\theta,\lambda}(z) \overset{def}{=} \frac{k_{\theta,\lambda}(z)}{\|k_{\theta,\lambda}(z)\|} = \left(\frac{1-|\lambda|^2}{1-|\theta(\lambda)|^2}\right)^{1/2} \frac{1-\overline{\theta(\lambda)}\theta(z)}{1-\overline{\lambda}z}$$

denotes the normalized reproducing kernel of K_θ. Let $\mathcal{B}(K_\theta)$ denote the algebra of all bounded linear operators on the space K_θ. In this article we shall investigate the following question: Let $A \in \mathcal{B}(K_\theta)$ satisfying

$$\left|\widetilde{A}^{K_\theta}(\lambda)\right| \geq \delta > 0 \ \ (\forall \lambda \in \mathbb{D})$$

for some $\delta > 0$. Under which conditions is A invertible in K_θ?

This question is closely related to a problem of Douglas [9] and works of Tolokonnikov, Nikolski, Wolff (see [24]) and the first author's paper [20]. Here, using the techniques of reproducing kernels and Berezin symbols, and an interpolation theorem of Shvedenko [27], we obtain sufficient conditions ensuring the invertibility of a linear bounded operators on the model space K_B with a suitable interpolation Blaschke product B (see Theorem 1 below). In particular, we investigate in terms of Berezin symbol the invertibility of some truncated Toeplitz operators (see Theorem 2 below). We also characterize in terms of Berezin symbols the normal operators on the Hardy space H^2, and study the compactness property of some products of Toeplitz operators on the Hardy and Bergman spaces.

2. Notations and Preliminaries

2.1. **Berezin symbol.** The Berezin symbol of a linear bounded operator T acting on a functional Hilbert space $\mathcal{H} = \mathcal{H}(\mathbb{D})$ over the unit disk $\mathbb{D}$, with a reproducing kernel $k_\lambda(z)$ is the complex-valued function

$$\widetilde{T}(\lambda) \overset{def}{=} \left\langle T\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle, \ \lambda \in \mathbb{D},$$

where $\widehat{k}_\lambda := k_\lambda / \|k_\lambda\|$ denotes the normalized reproducing kernel of $\mathcal{H}$. This notion has been introduced for the first time by Berezin [5, 6]. It is well known (see [11], [32]) that for a Toeplitz operator T_φ, with symbol $\varphi \in L^\infty(\mathbb{T})$, defined on H^2 by $T_\varphi f = P_+\varphi f$, where P_+ is the orthogonal projection from $L^2(\mathbb{T})$ onto H^2, known as the Riesz projection, its Berezin symbol $\widetilde{T_\varphi}$ is the harmonic extension $\widetilde{\varphi}$ of $\varphi \in L^\infty(\mathbb{T})$ into $\mathbb{D}$.

It is natural to define the following two numerical characteristics for the operator $T \in \mathcal{B}(\mathcal{H})$:

$$Ber(T) \overset{def}{=} Range(\widetilde{T}) \text{ is the so-called Berezin set}$$

and

$$ber(T) \overset{def}{=} \sup\{|\lambda| : \lambda \in Ber(T)\} \text{ is the so-called Berezin number.}$$

Obviously, $Ber(T) \subset W(T)$, where $W(T)$ is the numerical range of T. Also it is easy to see that $ber(T) \leq w(T)$, where $w(T)$ denotes the numerical radius of T.

In the case where $T = T_\varphi$, where φ is bounded, clearly $Ber(T_\varphi) = \{\widetilde{\varphi}(z) : z \in \mathbb{D}\}$ and $ber(T_\varphi) = \|\varphi\|_\infty$. In particular, $Ber(T_z) = \mathbb{D}$ and $ber(T_z) = 1$.

Note that on the most familiar functional Hilbert spaces, including the Hardy space and the Bergman space, the Berezin symbol uniquely determines the operator. In fact, if $\widetilde{T_1}(\lambda) = \widetilde{T_2}(\lambda)$ for all λ, then $T_1 = T_2$. See for instance, Yang [31]; and for more general cases, see Fricain [12, Theorem 1.1.1]. In other words the Berezin symbol of a bounded operator contains a lot of information about the operator. It is one of the most useful tools in the study of Toeplitz operators. The Berezin notion is motivated by its connections with quantum physics and noncommutative geometry. For more details and references see [5, 6]. Other properties and applications of Berezin symbols and reproducing kernels can be found in [2, 4, 7, 8, 13, 14, 15, 24].

2.2. **Carleson condition.** We now recall some well-known facts (see, for instance [24]) concerning reproducing kernels in $H^2(\mathbb{D})$. Let $\Lambda = \{\lambda_n\}_{n\geq 1}$ be a sequence of distinct points in $\mathbb{D}$. We denote by

$$B = B_\Lambda = \Pi_{n\geq 1} b_{\lambda_n},$$

where

$$b_{\lambda_n}(z) = \frac{|\lambda_n|}{\lambda_n}\frac{\lambda_n - z}{1-\overline{\lambda_n}z},$$

the corresponding Blaschke product. Then we have :

(i) If $\{\lambda_n\}_{n\geq 1}$ satisfies the Blaschke condition, i.e., $\sum_{n=1}^{\infty}\left(1-|\lambda_n|^2\right) < \infty$, then $\{k_{\lambda_n}\}_{n\geq 1}$ is a complete system in the model space K_B.

(ii) The family $\mathcal{K} \overset{def}{=} \left\{\widehat{k}_{\lambda_n} : n \geq 1\right\}$ is a Riesz basis of K_B if and only if $\{\lambda_n\}_{n\geq 1}$ satisfies the Carleson condition, namely

$$\inf_{n\geq 1} |B_n(\lambda_n)| > 0,$$

where $B_n \overset{def}{=} \frac{B}{b_{\lambda_n}}$. In this case, we will write $\Lambda \in (C)$.

2.3. **Riesz constant.** We recall (see [24]) that if H is a complex Hilbert space, and $\{x_n\}_{n\geq 1} \subset H$, then the set $X \overset{def}{=} \{x_n : n \geq 1\}$ is called a Riesz basis of H if there exists an isomorphism U mapping X onto an orthonormal basis of H. In this case the operator U will be called the orthogonalizer of X. It is well known (see [24]) that X is a Riesz basis in its closed linear span if there are two positive constants C_1, C_2 such that

$$C_1 \left(\sum_{n\geq 1} |a_n|^2\right)^{1/2} \leq \left\|\sum_{n\geq 1} a_n x_n\right\| \leq C_2 \left(\sum_{n\geq 1} |a_n|^2\right)^{1/2} \tag{1}$$

for all finite complex sequences $\{a_n\}_{n\geq 1}$. Note that if U is an orthogonalizer of the set X, then $C_1 = \| U \|^{-1}$ and $C_2 = \| U^{-1} \|$ are the best constants possible in the inequality (1). The product $r(X) \overset{def}{=} \|U\| \left\|U^{-1}\right\|$ characterizes the deviation of the basis X from an orthonormal one. $r(X)$ will be referred to as the Riesz constant of the family X. Clearly, $r(X) \geq 1$. For more detail, see [17, 24].

2.4. **Shvedenko constant.** Let $\{L_k\}_{k=1}^{\infty}$ be a sequence of linear continuous functionals on the Hardy space H^p, $1 < p < \infty$, (see Hoffman [18]). It is natural to try to describe the space of sequences

$$H^p\{L_k\} \overset{def}{=} \left\{\{L_k(f)\}_{k=1}^{\infty} : f \in H^p\right\}.$$

In particular, it is not without interest to try to find conditions under which the inclusion $S \subset H^p\{L_k\}$ is satisfied for a given space S of sequences of complex numbers. For some class of Banach spaces S of sequences, Shvedenko [27] gave a general criterion for such inclusion.

For $1 < p < \infty$, it is well known that the functionals L_k have the following representation

$$L_k(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(e^{it}\right) \overline{l_k\left(e^{it}\right)} dt, f \in H^p,$$

where the functions $l_k(z) \in H^q \left(\frac{1}{p} + \frac{1}{q} = 1\right)$ depend only on L_k. Notice that for the evaluation functionals

$$L_k(f) = f(\lambda_k), \ k = 1, 2, ...,$$

where $\{\lambda_k\}_{k=1}^{\infty}$ is a sequence of different points of $\mathbb{D}$, it is easy to see that $l_k(z) = \frac{1}{1-\overline{\lambda_k} z}$, which is the Szegö kernel.

Let S be a Banach space of sequences satisfying the following conditions

(S1) S be a BK-space [28], i.e., the map $w \to w_k, w = \{w_k\}_{k=1}^{\infty} \in S$, is continuous. This is equivalent to the inequality $|w_k| \leq c_k \|w\|_S$, where $\|w\|_S$ is the norm in S and $c_k > 0$, $k = 1, 2, ...$ In particular, this condition implies the inclusion $E_\infty \subset S^*$, where E_∞ is the space of sequences containing only a finite number of nonzero terms.

(S2) S is complexly conjugated, i.e., both of $w = \{w_k\}_{k=1}^{\infty}$ and $\overline{w} = \{\overline{w}_k\}_{k=1}^{\infty}$ belong to S and $\|\overline{w}\|_S = \|w\|_S$.

It is not difficult to verify that the classical weighted spaces $l^p(w_n)$, $p \geq 1$, satisfy conditions (S1) and (S2). The following key lemma is due to Shvedenko [27].

Lemma 1. *For the Banach spaces S of sequences satisfying conditions (S1) and (S2), the inclusion $S \subset H^p\{L_k\}$, $1 < p < \infty$, is fulfilled if and only if*

$$\inf_{<a_k> \in E_\infty} \frac{\left\|\sum_k a_k k_{\lambda_k}(z)\right\|_q}{\|< a_k >\|_{S^*}} > 0, \text{ where } \frac{1}{p} + \frac{1}{q} = 1.$$

In what follows we will call the number

$$\eta_\Lambda \overset{def}{=} \inf_{<a_k> \in E_\infty} \frac{\left\|\sum_k a_k k_{\lambda_k}(z)\right\|_q}{\|< a_k >\|_{S^*}}$$

the Shvedenko constant corresponding to the sequence $\Lambda \overset{def}{=} \{\lambda_k\}_{k=1}^{\infty}$.

It would be useful to note that Lemma 1 is simply the dual form of the condition that the embedding operator $J : (a_k) \to (a_k)$ form a sequence space S to the quotient space H^p/BH^p is bounded. This duality is well known and was systematically used in the interpolation theory, starting from seminal papers by Carleson, Shapiro-Shields, and Vinogradov-Havin on the 1960ies and 1970ies (see, for example, [24]). What is called the "Shvedenko constant" η_Λ is nothing but $1/C$, where $C = \|J^* : K_B^q \to S^*\|$. It is worth mentioning that the use of C as an interpolation constant is not allowed for $p \neq 2$ since the latter one is equal to $\|J\| = \left\|J^* : (H^p/BH^p)^* \to S^*\right\|$, whereas $(H^p/BH^p)^*$ is isomorphic but not isometric to $K_B^q = \left(L^p/BH^p + H_-^p\right)^*$ (excepting $p = 2$). We refer to the textbooks by Duren [10], Garnett [16] or Koosis [22] for details.

3. Invertibility of operators on the model space K_B

3.1. **General case.** The main result of this subsection is the following theorem.

Theorem 1. *Suppose that $\Lambda = \{\lambda_n\}_{n\geq 1}$ is a Carleson sequence of distinct points of $\mathbb{D}$, B is the interpolation Blaschke product associated to $\{\lambda_n\}_{n\geq 1}$, $\mathcal{K} = \left\{\widehat{k}_{\lambda_n} : n \geq 1\right\}$ is a corresponding Riesz basis of the space K_B, and $r(\mathcal{K})$ is the corresponding Riesz constant associated the family $\mathcal{K}$. Let $A \in \mathcal{B}(K_B)$ be any operator such that $\left|\widetilde{A}^{K_B}(\lambda n)\right| \geq \delta$ for all $n \geq 1$ and some $\delta > 0$. Let us denote*

$$\tau_A := \left(\sum_{n=1}^{\infty}\left(1-|\lambda_n|^2\right)\left\|\left(A-\widetilde{A}^{K_B}(\lambda_n)I\right)\widehat{k}_{\lambda_n}\right\|^2\right)^{1/2}$$

and

$$\tau_A^* := \left(\sum_{n=1}^{\infty}\left(1-|\lambda_n|^2\right)\left\|\left(A-\widetilde{A}^{K_B}(\lambda_n)I\right)^*\widehat{k}_{\lambda_n}\right\|^2\right)^{1/2}.$$

If

$$\delta > \frac{r(\mathcal{K})}{\eta_\Lambda}\max\{\tau_A, \tau_A^*\},$$

then A is an invertible operator in K_B. Moreover

$$\left\|A^{-1}\right\| \leq \frac{r(\mathcal{K})\,\eta_\Lambda}{\delta\eta_\Lambda - r(\mathcal{K})\,\tau_A},$$

where η_Λ is the Shvedenko constant corresponding to the sequence $\Lambda = \{\lambda_n\}_{n\geq 1}$.

Proof. Since $\left\{\left\|A\widehat{k}_{\lambda_n} - \widetilde{A}^{K_B}(\lambda_n)\widehat{k}_{\lambda_n}\right\|\right\}_{n\geq 1}$ and $\left\{\left\|A^*\widehat{k}_{\lambda_n} - \overline{\widetilde{A^*}^{K_B}(\lambda_n)}\widehat{k}_{\lambda_n}\right\|\right\}_{n\geq 1}$ are bounded sequences and Λ is a Blaschke sequence, the numbers τ_A and τ_A^* are finite. Also, the family $\mathcal{K} = \left\{\widehat{k}_{\lambda_n} : n \geq 1\right\}$ is a Riesz basis in K_B, because $\Lambda \in (C)$. If U is an orthogonalizer of $\mathcal{K}$, then

$$\|U\|^{-1}\left(\sum_{n\geq 1}|a_n|^2\right)^{1/2} \leq \left\|\sum_{n\geq 1}a_n\widehat{k}_{\lambda_n}\right\| \leq \left\|U^{-1}\right\|\left(\sum_{n\geq 1}|a_n|^2\right)^{1/2} \tag{2}$$

for any finite complex sequence $\{a_n\}_{n\geq 1}$. Hence by considering (2) and the condition $\left|\widetilde{A}^{K_B}(\lambda n)\right| \geq \delta > 0$, $n \geq 1$, we have that for any $N > 0$

$$\begin{aligned}\left\|\sum_{n=1}^{N}a_n\widetilde{A}^{K_B}(\lambda_n)\widehat{k}_{\lambda_n}\right\| &\geq \|U\|^{-1}\left(\sum_{n=1}^{N}\left|a_n\widetilde{A}^{K_B}(\lambda_n)\right|^2\right)^{1/2}\\ &\geq \delta\|U\|^{-1}\left(\sum_{n=1}^{N}|a_n|^2\right)^{1/2}\\ &\geq \frac{\delta}{\|U\|\left\|U^{-1}\right\|}\left\|\sum_{n=1}^{N}a_n\widehat{k}_{\lambda_n}\right\|,\end{aligned}$$

or

$$\left\|\sum_{n=1}^{N}a_n\widetilde{A}^{K_B}(\lambda_n)\widehat{k}_{\lambda_n}\right\| \geq \frac{\delta}{r(\mathcal{K})}\left\|\sum_{n=1}^{N}a_n\widehat{k}_{\lambda_n}\right\|. \tag{3}$$

Now

$$\begin{aligned}\left\| A\sum_{n=1}^{N} a_n \widehat{k}_{\lambda_n}\right\| &= \left\| \sum_{n=1}^{N} a_n A\widehat{k}_{\lambda_n}\right\| \\ &= \left\| \sum_{n=1}^{N} a_n \left(A\widehat{k}_{\lambda_n} - \widetilde{A}^{K_B}(\lambda_n)\widehat{k}_{\lambda_n} + \widetilde{A}^{K_B}(\lambda_n)\widehat{k}_{\lambda_n}\right)\right\| \\ &\geq \left\| \sum_{n=1}^{N} a_n \widetilde{A}^{K_B}(\lambda_n)\widehat{k}_{\lambda_n}\right\| - \sum_{n=1}^{N} |a_n| \left\| A\widehat{k}_{\lambda_n} - \widetilde{A}^{K_B}(\lambda_n)\widehat{k}_{\lambda_n}\right\|.\end{aligned}$$

It follows from inequality (3) that

$$\begin{aligned}\left\| A\sum_{n=1}^{N} a_n \widehat{k}_{\lambda_n}\right\| &\geq \frac{\delta}{r(\mathcal{K})}\left\| \sum_{n=1}^{N} a_n \widehat{k}_{\lambda_n}\right\| \\ &\quad -\sum_{n=1}^{N} \frac{|a_n|}{\left(1-|\lambda_n|^2\right)^{1/2}}\left(1-|\lambda_n|^2\right)^{1/2}\left\| A\widehat{k}_{\lambda_n} - \widetilde{A}^{K_B}(\lambda_n)\widehat{k}_{\lambda_n}\right\|.\end{aligned}$$

Using Holder inequality, we obtain

$$\begin{aligned}\left\| A\sum_{n=1}^{N} a_n \widehat{k}_{\lambda_n}\right\| &\geq \frac{\delta}{r(\mathcal{K})}\left\| \sum_{n=1}^{N} a_n \widehat{k}_{\lambda_n}\right\| \\ &\quad -\left(\sum_{n=1}^{N} \frac{|a_n|^2}{\left(1-|\lambda_n|^2\right)}\right)^{1/2}\left(\sum_{n=1}^{N}\left(1-|\lambda_n|^2\right)\left\| A\widehat{k}_{\lambda_n} - \widetilde{A}^{K_B}(\lambda_n)\widehat{k}_{\lambda_n}\right\|^2\right)^{1/2} \\ &\geq \frac{\delta}{r(\mathcal{K})}\left\| \sum_{n=1}^{N} a_n \widehat{k}_{\lambda_n}\right\| - \left(\sum_{n=1}^{N}\frac{|a_n|^2\left(1-|\lambda_n|^2\right)}{\left(1-|\lambda_n|^2\right)^2}\right)^{1/2}\tau_A.\end{aligned}$$

Writing that

$$\left(\sum_{n=1}^{N}\frac{|a_n|^2\left(1-|\lambda_n|^2\right)}{\left(1-|\lambda_n|^2\right)^2}\right)^{1/2} = \left\|\left\{a_n(1-|\lambda_n|^2)^{1/2}\right\}_{n=1}^{N}\right\|_{l^2\left((1-|\lambda_n|^2)^{-1}\right)},$$

we have

$$\left\| A\sum_{n=1}^{N} a_n \widehat{k}_{\lambda_n}\right\| \geq \frac{\delta}{r(\mathcal{K})}\left\| \sum_{n=1}^{N} a_n \widehat{k}_{\lambda_n}\right\| - \left\|\left\{a_n(1-|\lambda_n|^2)^{1/2}\right\}_{n=1}^{N}\right\|_{l^2\left((1-|\lambda_n|^2)^{-1}\right)}\eta_\Lambda\eta_\Lambda^{-1}\tau_A,$$

where η_Λ is the Shvedenko constant for Λ. Because $\{\lambda_n\}_{n\geq 1}$ is assumed to be a Carleson sequence, it is well-known in this case that

$$H^2\{L_n\} = l^2\left(1-|\lambda_n|^2\right).$$

Thus, by considering the obvious inclusion $l^2\left(\left(1-|\lambda_n|^2\right)^{-1}\right) \subset l^2\left(1-|\lambda_n|^2\right)$, we have that $l^2\left(\left(1-|\lambda_n|^2\right)^{-1}\right) \subset H^2\{L_n\}$. Therefore, by setting $p=2$ and $S = l^2\left(\left(1-|\lambda_n|^2\right)^{-1}\right)$ in Lemma 1, we obtain

$$\begin{aligned} \left\| A\sum_{n=1}^{N} a_n \widehat{k}_{\lambda_n} \right\| &\geq \frac{\delta}{r(\mathcal{K})}\left\| \sum_{n=1}^{N} a_n \widehat{k}_{\lambda_n} \right\| - \left\| \sum_{n=1}^{N} a_n \left(1-|\lambda_n|^2\right)^{1/2} k_{\lambda_n} \right\| \eta_\Lambda^{-1}\tau_A \\ &= \frac{\delta}{r(\mathcal{K})}\left\| \sum_{n=1}^{N} a_n \widehat{k}_{\lambda_n} \right\| - \eta_\Lambda^{-1}\tau_A \left\| \sum_{n=1}^{N} a_n \widehat{k}_{\lambda_n} \right\|. \end{aligned}$$

Finally, we arrive to the following inequality

$$\left\| A\sum_{n=1}^{N} a_n \widehat{k}_{\lambda_n} \right\| \geq \left(\frac{\delta}{r(\mathcal{K})} - \frac{\tau_A}{\eta_\Lambda}\right)\left\| \sum_{n=1}^{N} a_n \widehat{k}_{\lambda_n} \right\| \tag{4}$$

for any complex numbers a_n, $n = \overline{1,N}$, and all $N > 0$. Since the Carleson condition implies the Blaschke condition, $\mathrm{Span}(\mathcal{K}) = K_B$, i.e., $\mathcal{K}$ is a complete system in K_B. Therefore, we deduce from (4) that

$$\|Af\| \geq \left(\frac{\delta}{r(\mathcal{K})} - \frac{\tau_A}{\eta_\Lambda}\right)\|f\|, \tag{5}$$

for any $f \in K_B$.

By similar arguments, we prove that

$$\left\| A^*\sum_{n=1}^{N} a_n \widehat{k}_{\lambda_n} \right\| \geq \left(\frac{\delta}{r(\mathcal{K})} - \frac{\tau_A^*}{\eta_\Lambda}\right)\left\| \sum_{n=1}^{N} a_n \widehat{k}_{\lambda_n} \right\|,$$

which yields that

$$\|A^*f\| \geq \left(\frac{\delta}{r(\mathcal{K})} - \frac{\tau_A^*}{\eta_\Lambda}\right)\|f\| \tag{6}$$

for any $f \in K_B$. Now, combining hypothesis (**H**) with both inequalities (5) and (6), implies that A is invertible in K_B and that

$$\left\|A^{-1}\right\| \leq \frac{r(\mathcal{K})\,\eta_\Lambda}{\eta_\Lambda\delta - r(\mathcal{K})\,\tau_A},$$

which completes the proof. □

3.2. **Truncated Toeplitz operators.** Consider the truncated Toeplitz operator $\mathcal{T}_\varphi$ acting on the subspace K_θ in the following way

$$\mathcal{T}_{\varphi,\theta} f := P_\theta T_\varphi f, \;\; f \in K_\theta,$$

where T_θ is the Toeplitz operator in the Hardy space H^2 defined by $T_\varphi f = P_+ \varphi f$, $\varphi \in L^\infty(\mathbb{T})$ and θ is an inner function.

In this subsection we will study the invertibility property of the truncated Toeplitz operator $\mathcal{T}_{\varphi,\theta}$, which is closely related with Problem 2 posed by the first author in [20]. As an application of Theorem 1, we prove here the following theorem which partially solves Problem 2 in [20]. As a corollary of this result, we obtain that the so-called "Tolokonnikov-Nikolski conditions"

$$1 \geq |\widetilde{\varphi}(z)| \geq \delta > \frac{45}{46} \ \ (z \in \mathbb{D})$$

and

$$1 \geq |\widetilde{\varphi}(z)| \geq \delta > \frac{23}{24} \ \ (z \in \mathbb{D})$$

for the invertibility of Toeplitz operators on H^2 provide also invertibility of some truncated Toeplitz operators (see, Corollary 2 below).

The main result of this subsection is the following theorem.

Theorem 2. *Let $\Lambda = \{\lambda_n\}_{n\geq 1}$, η_Λ, B and $r(\mathcal{K})$ be as in Theorem 1. Let $\varphi \in L^\infty(\mathbb{T})$ be a function satisfying $\|\varphi\|_{L^\infty} \leq 1$, for which there exists a constant $\delta > 0$ such that*

$$|\widetilde{\varphi}(\lambda_n)| \geq \delta > \sqrt{\frac{r(\mathcal{K})^2 w_\Lambda^2}{\eta_\Lambda^2 + r(\mathcal{K})^2 w_\Lambda^2}}, \ \textit{for all } n \geq 1, \tag{7}$$

where $w_\Lambda := \left(\sum_{n=1}^\infty \left(1 - |\lambda_n|^2\right)\right)^{1/2}$. Then the truncated Toeplitz operator $\mathcal{T}_{\varphi,B} := P_B T_\varphi | K_B$ is invertible and

$$\left\|(\mathcal{T}_{\varphi,B})^{-1}\right\| \leq \left[\frac{\delta}{r(\mathcal{K})} - \frac{w_\Lambda}{\eta_\Lambda}\left(1-\delta^2\right)^{1/2}\right]^{-1}.$$

Proof. Considering that

$$\widehat{k}_{B,\lambda}(z) = \left(\frac{1-|z|^2}{1-|B(\lambda)|^2}\right)^{1/2} \frac{1-\overline{B(\lambda)}B(z)}{1-\overline{\lambda}z}$$

is the normalized reproducing kernel for the subspace K_B, it is easy to verify that

$$\widetilde{\mathcal{T}}_{\varphi,B}^{K_B}(\lambda_n) = \widetilde{\varphi}(\lambda_n)$$

for all $n \geq 1$, where $\widetilde{\varphi} = \widetilde{T}_{\varphi}$ is the harmonic extension of $\varphi \in L^{\infty}(\mathbb{T})$ into $\mathbb{D}$. Moreover, using the fact that $P_B \widehat{k}_{\lambda_n} = \widehat{k}_{\lambda_n}$, we have

$$\begin{aligned}
\left\| \mathcal{T}_{\varphi,B} \widehat{k}_{\lambda_n} - \widetilde{\varphi}(\lambda_n) \widehat{k}_{\lambda_n} \right\|^2 &= \left\langle P_B T_{\varphi} \widehat{k}_{\lambda_n}, P_B T_{\varphi} \widehat{k}_{\lambda_n} \right\rangle - \overline{\widetilde{\varphi}(\lambda_n)} \left\langle P_B T_{\varphi} \widehat{k}_{\lambda_n}, \widehat{k}_{\lambda_n} \right\rangle \\
&\quad - \widetilde{\varphi}(\lambda_n) \left\langle \widehat{k}_{\lambda_n}, P_B T_{\varphi} \widehat{k}_{\lambda_n} \right\rangle + \left| \widetilde{\varphi}(\lambda_n) \right|^2 \\
&= \widetilde{T_{\overline{\varphi}} P_B T_{\varphi}}(\lambda_n) - \overline{\widetilde{\varphi}(\lambda_n)} \left\langle T_{\varphi} \widehat{k}_{\lambda_n}, \widehat{k}_{\lambda_n} \right\rangle \\
&\quad - \widetilde{\varphi}(\lambda_n) \left\langle \widehat{k}_{\lambda_n}, T_{\varphi} \widehat{k}_{\lambda_n} \right\rangle + \left| \widetilde{\varphi}(\lambda_n) \right|^2 \\
&= \widetilde{T_{\overline{\varphi}} P_B T_{\varphi}}(\lambda_n) - \left| \widetilde{\varphi}(\lambda_n) \right|^2 \\
&\leq ber\left(T_{\overline{\varphi}} P_B T_{\varphi}\right) - \delta^2 \\
&\leq \left\| T_{\overline{\varphi}} P_B T_{\varphi} \right\| - \delta^2 \\
&\leq \left\| \varphi \right\|_{L^{\infty}}^2 - \delta^2 \\
&\leq 1 - \delta^2.
\end{aligned}$$

Thus

$$\left\| \mathcal{T}_{\varphi,B} \widehat{k}_{\lambda_n} - \widetilde{\mathcal{T}}_{\varphi,B}^{K_B}(\lambda_n) \widehat{k}_{\lambda_n} \right\|^2 \leq 1 - \delta^2, \ n \geq 1. \tag{8}$$

Inequality (8) implies that

$$\mathcal{T}_{\mathcal{T}_{\varphi,B}} \leq w_{\Lambda} \left(1 - \delta^2\right)^{1/2}.$$

Now, considering that $\mathcal{T}_{\varphi,B}^{*} = P_B T_{\varphi}^{*} | K_B = P_B T_{\overline{\varphi}} | K_B$ and $\widetilde{T}_{\overline{\varphi}} = \overline{\widetilde{T}}_{\varphi} = \overline{\widetilde{\varphi}}$, and using the same argument as in the proof of Theorem 1, one can show that

$$\left\| \mathcal{T}_{\varphi,B} f \right\| \geq \left[\frac{\delta}{r(\mathcal{K})} - \frac{w_{\Lambda}}{\eta_{\Lambda}} \left(1 - \delta^2\right)^{1/2} \right] \left\| f \right\|$$

and

$$\left\| \mathcal{T}_{\varphi,B}^{*} f \right\| \geq \left[\frac{\delta}{r(\mathcal{K})} - \frac{w_{\Lambda}}{\eta_{\Lambda}} \left(1 - \delta^2\right)^{1/2} \right]^{-1} \left\| f \right\|$$

for every $f \in K_B$. Therefore, Theorem 1 implies that $\mathcal{T}_{\varphi,B}$ is invertible, and

$$\left\| \mathcal{T}_{\varphi,B}^{-1} \right\| \leq \left[\frac{\delta}{r(\mathcal{K})} - \frac{w_{\Lambda}}{\eta_{\Lambda}} \left(1 - \delta^2\right)^{1/2} \right],$$

which completes the proof, because the condition $\delta > r(\mathcal{K}) w_{\Lambda} \sqrt{\frac{1}{\eta_{\Lambda}^2 + r(\mathcal{K})^2 w_{\Lambda}^2}}$ is equivalent to the inequality

$$\frac{\delta}{r(\mathcal{K})} - \frac{w_{\Lambda}}{\eta_{\Lambda}} \left(1 - \delta^2\right)^{1/2} > 0.$$

□

Remark 1. *One of the important special classes of truncated Toeplitz operators is the class of model operators $\varphi(M_\theta)$ of Sz.-Naggy and Foias defined for each $\varphi \in H^\infty$ by the formula*

$$\varphi(M_\theta) f = P_\theta \varphi f, \; f \in K_\theta,$$

where θ is an inner function and $P_\theta = I - T_\theta T_{\overline{\theta}}$ is the orthogonal projection from H^2 onto $K_\theta = H^2 \ominus \theta H^2$. For such operators, the invertibility problem is solved by means of the celebrated Carleson Corona Theorem [24] *that the model operator $\varphi(M_\theta)$ is invertible if and only if there exists a constant $\delta > 0$ such that*

$$|\varphi(z)| + |\theta(z)| \geq \delta \tag{9}$$

for all $z \in \mathbb{D}$.

When θ is an interpolation Blaschke product B, i.e., Blaschke product with zeros $\Lambda = \{\lambda_n\}_{n\geq 1} \in (C)$, it is also known (see, for instance, Hoffman [18, Chapter 10]) that condition (9) *is equivalent to*

$$|\varphi(\lambda_n)| \geq \delta \;\; (n \geq 1). \tag{10}$$

It is relevant to note that under last condition (10), *for the invertibility of operator $\varphi(M_B)$ there is an elementary proof which does not depend neither corona nor interpolation theorems. Indeed, if one supposes that the sequences $\left\{\widehat{k}_{\lambda_n}\right\}_{n\geq 1}$ is a Riesz basis (as in Theorem 2), then it is obvious that an operator $\varphi(M_B)$, $\varphi \in H^\infty$ (for which $\varphi(M_B)^* \widehat{k}_{\lambda_n} = T_{\overline{\varphi}} \widehat{k}_{\lambda_n} = \overline{\varphi(\lambda_n)} \widehat{k}_{\lambda_n}$ for every $n \geq 1$), is invertible if and only if $\delta := \inf_{n\geq 1} |\varphi(\lambda_n)| > 0$, and if this is the case, we have by means of inequality* (2) *that $\left\| \varphi(M_\theta)^{-1} \right\| \leq \frac{r(\mathcal{K})}{\delta}$.*

Remark 2. *If $\Lambda = \{\lambda_n\} \in (C)$ is a sequence such that*

$$\eta_\Lambda^2 + r(\mathcal{K})^2 w_\Lambda^2 = (r(\mathcal{K}) w_\Lambda + 1)^2$$

(i.e., if $\eta_\Lambda = \sqrt{1 + 2r(\mathcal{K}) w_\Lambda}$), then condition (7) *in Theorem 2, becomes*

$$1 \geq |\widetilde{\varphi}(\lambda_n)| \geq \delta > \frac{r(\mathcal{K}) w_\Lambda}{r(\mathcal{K}) w_\Lambda + 1}. \tag{11}$$

In particular, if $r(\mathcal{K}) w_\Lambda = 45$, then $\eta_\Lambda = \sqrt{91}$. Therefore condition (11) *becomes*

$$1 \geq |\widetilde{\varphi}(\lambda_n)| \geq \delta > \frac{45}{46}, \text{ which is a "Tolokonnikov type" condition [30].}$$

If $r(\mathcal{K}) w_\Lambda = 23$, we obtain that

$$1 \geq |\widetilde{\varphi}(\lambda_n)| \geq \delta > \frac{23}{24}, \text{ which is a "Nikolski type" condition [24].}$$

(More details about "Tolokonnikov-Nikolski type" invertibility conditions for the Toeplitz operators on the Hardy space H^2 can be found in [24]*.)*

This remark shows that the following corollary of Theorem 2 is true.

Corollary 1. *We have:*

(a) If $\delta > \frac{45}{46}$, then $\mathcal{T}_{\varphi,B}$ is invertible and

$$\left\|\mathcal{T}_{\varphi,B}^{-1}\right\| \leq \frac{\sqrt{91}r\left(\mathcal{K}\right)}{\sqrt{91}\delta - 45\sqrt{1-\delta^2}}.$$

(b) If $\delta > \frac{23}{24}$, then $\mathcal{T}_{\varphi,B}$ is invertible and

$$\left\|\mathcal{T}_{\varphi,B}^{-1}\right\| \leq \frac{\sqrt{47}r\left(\mathcal{K}\right)}{\sqrt{47}\delta - 23\sqrt{1-\delta^2}}.$$

4. Further Results

4.1. Normal and Toeplitz operators on H^2.

We shall characterize normal operators on the Hardy space H^2 in terms of Berezin symbols. Also, we shall discuss compactness properties of products of some Toeplitz operators acting on Hardy and Bergman spaces.

Theorem 3. *Let A be a bounded operator on H^2, and let $\widetilde{A}\left(\lambda\right) = \left\langle A\widehat{k}_{\lambda}\left(z\right), \widehat{k}_{\lambda}\left(z\right)\right\rangle$ be its Berezin symbol, where $\widehat{k}_{\lambda}\left(z\right) = \frac{\sqrt{1-|\lambda|^2}}{1-\overline{\lambda}z}$ is a normalized reproducing kernel of H^2. Then*

(i) *A is a normal operator on H^2 if and only if*

$$\left\|\left(A - \widetilde{A}\left(\lambda\right)I\right)\widehat{k}_{\lambda}\right\| = \left\|\left(A - \widetilde{A}\left(\lambda\right)I\right)^*\widehat{k}_{\lambda}\right\| \text{ for all } \lambda \in \mathbb{D}.$$

(ii) *In particular, if $A = T_{\varphi}$, where $\varphi \in L^{\infty}$, then the product $T_{\varphi}^*T_{\varphi}$ $\left(\text{or } T_{\varphi}T_{\varphi}^*\right)$ is compact if and only if $\varphi = 0$.*

Proof. An easy computation shows that

$$\left\|A\widehat{k}_{\lambda} - \widetilde{A}\left(\lambda\right)\widehat{k}_{\lambda}\right\|^2 = \widetilde{A^*A}\left(\lambda\right) - \left|\widetilde{A}\left(\lambda\right)\right|^2 \tag{12}$$

and

$$\left\|A^*\widehat{k}_{\lambda} - \widetilde{A^*}\left(\lambda\right)\widehat{k}_{\lambda}\right\|^2 = \widetilde{AA^*}\left(\lambda\right) - \left|\widetilde{A}\left(\lambda\right)\right|^2. \tag{13}$$

Since the Berezin symbol uniquely determines the operator A, it follows from formulas (12), (13) that A is a normal operator on H^2 if and only if

$$\left\|\left(A - \widetilde{A}\left(\lambda\right)I\right)\widehat{k}_{\lambda}\right\| = \left\|\left(A - \widetilde{A}\left(\lambda\right)I\right)^*\widehat{k}_{\lambda}\right\| \ \left(\forall\lambda \in \mathbb{D}\right),$$

which proves (i).

On the other hand, it is known that (see Engliś [11] and Karaev [21])

$$\left\|T_{\varphi}\widehat{k}_{\lambda} - \widetilde{\varphi}\left(\lambda\right)\widehat{k}_{\lambda}\right\| \to 0 \text{ as } \lambda \to \mathbb{T} \text{ radially}.$$

In fact, the functions $\widehat{k}_\lambda$ $(\lambda \in \mathbb{D})$ are "loosely speaking" asymptotic eigenfunctions for the Toeplitz operator T_φ, with asymptotic eigenvalues $\widetilde{\varphi}(\lambda)$. Since $\widetilde{T}_\varphi(\lambda) = \widetilde{\varphi}(\lambda)$ is the harmonic extension of φ onto unit disk $\mathbb{D}$, it follows from equalities (12) and (13) that

$$\left\| T_\varphi \widehat{k}_\lambda - \widetilde{\varphi}(\lambda)\widehat{k}_\lambda \right\|^2 = \widetilde{T_\varphi^* T_\varphi}(\lambda) - |\widetilde{\varphi}(\lambda)|^2 \tag{14}$$

and

$$\left\| T_\varphi^* \widehat{k}_\lambda - \widetilde{\overline{\varphi}}(\lambda)\widehat{k}_\lambda \right\|^2 = \widetilde{T_\varphi T_\varphi^*}(\lambda) - |\widetilde{\varphi}(\lambda)|^2 . \tag{15}$$

Now, using the fact that the Berezin symbol of any compact operator on H^2 vanishes at the boundary $\mathbb{T}$, and considering the above mentioned facts, we deduce from (14) and (15) that if $T_\varphi^* T_\varphi$ (or $T_\varphi T_\varphi^*$) is a compact operator on H^2 then

$$\lim_{r \to 1^-} \left|\widetilde{\varphi}\left(re^{it}\right)\right|^2 = \left|\varphi\left(e^{it}\right)\right|^2 = 0$$

for almost all $t \in [0, 2\pi)$, and hence $\varphi = 0$. This proves (ii), and the theorem is then proved. □

4.2. **The Bergman space Toeplitz operators.** In the Bergman space, as usual, things are much more complicated. Assertion (ii) of Theorem 3 is not true for the Bergman space Toeplitz operators. In Example 1 below , we were able to find a nonzero radial symbol f such that the product T_f^2 is equal to a compact Toeplitz operator T_g.

Let $dA = rdr\frac{d\theta}{\pi}$, where (r, θ) are the polar coordinates in the complex plane $\mathbb{C}$, denote the normalized Lebesgue area measure on the unit disk $\mathbb{D}$, so that the measure of $\mathbb{D}$ equals 1. The Bergman space $L_a^2(\mathbb{D})$ is the Hilbert space consisting of the analytic functions on $\mathbb{D}$ that are also square integrable with respect to the measure dA. We denote the inner product in $L^2(\mathbb{D}, dA)$ by $\langle , \rangle$. It is well known that $L_a^2(\mathbb{D})$ is a closed subspace of the Hilbert space $L^2(\mathbb{D}, dA)$, and has the set $\{\sqrt{n+1}z^n \mid n \geq 0\}$ as an orthonormal basis. We let P be the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto $L_a^2(\mathbb{D})$. For a bounded function f on $\mathbb{D}$, the Toeplitz operator T_f with symbol f is defined by

$$T_f(h) = P(fh) \text{ for } h \in L_a^2(\mathbb{D}).$$

It is well known that if the symbol f is a radial function, i.e. $f(z) = f(|z|)$, then the matrix of the Toeplitz operator T_f, with respect to the orthonormal basis $\{\sqrt{n+1}z^n \mid n \geq 0\}$ of $L_a^2(\mathbb{D})$, is a diagonal matrix with the sequence

$\left\{2(n+1)\int_0^1 f(r)r^{2n+1}\,dr\right\}_{n\geq 0}$ as elements of the main diagonal. In fact

$$\begin{aligned}\langle T_f(\sqrt{n+1}z^n),\sqrt{m+1}z^m\rangle &= \langle P(f\sqrt{n+1}z^n),\sqrt{m+1}z^m\rangle\\ &= \sqrt{n+1}\sqrt{m+1}\langle fz^n,z^m\rangle\\ &= \sqrt{n+1}\sqrt{m+1}\int_0^1\int_0^{2\pi} f(r)r^{n+m+1}e^{i(n-m)\theta}\frac{d\theta}{2\pi}dr\\ &= \begin{cases} 2(n+1)\displaystyle\int_0^1 f(r)r^{2n+1}\,dr & \text{if } n=m\\ 0 & \text{if } n\neq m\end{cases}\end{aligned}$$

Moreover, it has been shown in [23, Proposition 4.3 p.530], that the product T_f^2 is equal to a Toeplitz operator T_g if and only if there exists a radial symbol g solution to the following Mellin convolution equation

$$\int_r^1 g(t)\frac{dt}{t} = \int_r^1 f\left(\frac{r}{t}\right)f(t)\frac{dt}{t}. \tag{16}$$

Now we are ready to present our counterexample to condition (ii) of Theorem 3 in the case of Bergman space Toeplitz operators.

Example 1. *Let $f(r) = r\ln r$. By solving equation (16) for g, we obtain*

$$g(r) = \frac{r}{2}\left(\frac{1}{3}\ln r - 1\right)(\ln r)^2.$$

Hence $T_f^2 = T_g$. Obviously f and g are not bounded but they are the so-called "nearly bounded functions" [1, p.204]. Thus the Toeplitz operators associated to these two symbols are bounded. Since g is a radial symbol, T_g is a diagonal operator with the sequence $\left\{2(n+1)\int_0^1 g(r)r^{2n+1}dr\right\}_{n\geq 0}$ as elements of the main diagonal. In this case, it is well known that T_g will be compact if and only if

$$\lim_{n\to+\infty} 2(n+1)\left|\int_0^1 g(r)r^{2n+1}dr\right| = 0.$$

Now since $T_g = T_f^2$, a direct calculation shows that

$$\begin{aligned}2(n+1)\int_0^1 g(r)r^{2n+1}dr &= \left(2(n+1)\int_0^1 f(r)r^{2n+1}dr\right)^2\\ &= \frac{4(n+1)^2}{(2n+3)^4},\ \textit{for all } n\geq 0.\end{aligned}$$

It is clear that the fraction above will tend to zero as n goes to infinity. Hence T_g, and therefore T_f^2 is compact. But f is not the zero function.

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Mübariz T. Karaev, Suleyman Demirel University Isparta Vocational School 32260 Isparta, Turkey

E-mail address: garayev@fef.sdu.edu.tr

Issam Louhichi, King Fahd University of Petroleum & Minerals Department of Mathematics & Statistics Dhahran 31261, Saudi Arabia

E-mail address: issam@kfupm.edu.sa