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Complex Analysis and Operator Theory

Finite Rank Commutators and Semicommutators of Quasihomogeneous Toeplitz Operators

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Abstract. We study finite rank semicommutators and commutators of Toeplitz operators on the Bergman space with quasihomogeneous symbols. We show that in this context, the situation is different from the case of harmonic Toeplitz operators.

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1. Introduction

Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$, and $dA = rdr\frac{d\theta}{\pi}$ be the normalized Lebesgue area measure so that the measure of $\mathbb{D}$ equals 1. The Bergman space L_a^2 is the Hilbert space consisting of the analytic functions on $\mathbb{D}$ that are also square integrable with respect to the measure dA. We denote the inner product in $L^2(\mathbb{D}, dA)$ by $\langle , \rangle$. It is well known that L_a^2 is a closed subspace of the Hilbert space $L^2(\mathbb{D}, dA)$, and has the set $\{\sqrt{n+1}z^n \mid n \geq 0\}$ as an orthonormal basis. We let P be the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto L_a^2. For a bounded function ϕ on $\mathbb{D}$, the Toeplitz operator T_ϕ with symbol ϕ is defined by

$$T_\phi(h) = P(\phi h) \quad \text{for} \quad h \in L_a^2 .$$

In the last two decades, a lot of work has been done in understanding the algebraic properties of Toeplitz operators on the Bergman space. This includes studying the semicommutators and commutators of Toeplitz operators. For two Toeplitz operators T_ϕ and T_ψ we define the semicommutator and the commutator respectively by

$$(T_\phi, T_\psi] = T_{\phi\psi} - T_\phi T_\psi$$

and

$$[T_\phi, T_\psi] = T_\phi T_\psi - T_\psi T_\phi \,.$$

Commuting Toeplitz operators with harmonic symbols were characterized by Axler and the first author [4], and essentially commuting by Stroethoff [16]. The zero semicommutators were studied by Zheng [18]. The natural question to ask is when these objects are compact or finite rank. In this respect we should mention here the characterization of compact Toeplitz operators by Axler and Zheng [5] in terms of the Berezin transform of the symbol. In this context we also mention the work of Suárez [17]. Very recently, Luecking [14] has proved that the only finite rank Toeplitz operator is the zero operator. Finite rank commutators and semicommutators of Toeplitz operators with harmonic symbols were characterized by Guo, Sun and Zheng [10]. About the same time, finite rank perturbations of related products of Toeplitz operators were studied by the first author [7]. Both results in [10] and [7] were generalized in a recent paper by Choe, Koo and Lee [6].

In this paper we continue this line of investigation and study semicommutators and commutators of quasihomogeneous Toeplitz operators. A function ϕ is said to be quasihomogeneous of degree p if it is of the form $e^{ip\theta} f$, where f is a radial function. In this case the associated Toeplitz operator T_ϕ is also called quasihomogeneous Toeplitz operator of degree p. These Toeplitz operators were studied in [8, 11, 12] and [13]. The reason that we study such family of symbols is that any function f in $L^2(\mathbb{D}, dA)$ has the following polar decomposition

$$f(re^{ik\theta}) = \sum_{k\in\mathbb{Z}} e^{ik\theta} f_k(r)\,,$$

where f_k are radial functions in $L^2([0,1], rdr)$. Unlike the harmonic case where finite rank semicommutators and commutators are zero, we show that semicommutators and commutators of two quasihomogeneous Toeplitz operators of opposite degrees can be nonzero finite rank operators. Our techniques are very different from those used in the harmonic case. We use the Mellin transform instead of the Berezin transform. Moreover we support our results by giving an effective constuction of nontrivial finite rank semicommutators and commutators of quasihomogeneous Toeplitz operators. Finding examples of nontrivial Toeplitz operators satisfying certain algebraic properties has been notoriously difficult, but such constructions are now possible using recent results due to the work of the first author and Rao [8], the second author, Zakariasy and Strouse [11] and the second author and Zakariasy [12].

2. Preliminaries

Before we state our results, we need to introduce the Mellin transform which is going to be our main tool. The Mellin transform $\widehat{f}$ of a radial function f in $L^1([0,1], rdr)$ is defined by

$$\widehat{f}(z) = \int_0^1 f(r) r^{z-1}\, dr\,.$$

It is well known that, for these functions, the Mellin transform is well defined on the right half-plane $\{z : \Re z \geq 2\}$ and it is analytic on $\{z : \Re z > 2\}$. It is important and helpful to know that the Mellin transform $\widehat{f}$ is uniquely determined by its values on an arithmetic sequence of integers. In fact we have the following classical theorem [15, p. 102].

Theorem 1. *Suppose that f is a bounded analytic function on $\{z : \Re z > 0\}$ which vanishes at the pairwise distinct points $z_1, z_2, \ldots$, where*

i) $\inf\{|z_n|\} > 0$ *and*
ii) $\sum_{n\geq 1} \Re(\frac{1}{z_n}) = \infty$.

Then f vanishes identically on $\{z : \Re z > 0\}$.

Remark 1. Now one can apply this theorem to prove that if $f \in L^1([0,1], rdr)$ and if there exist $n_0, p \in \mathbb{N}$ such that

$$\widehat{f}(pk + n_0) = 0 \quad \text{for all} \quad k \in \mathbb{N}\,,$$

then $\widehat{f}(z) = 0$ for all $z \in \{z : \Re z > 2\}$ and so $f = 0$.

The use of the Mellin transform in the study of Toeplitz operators was introduced for the first time by the first author and Rao in [8]. They used it to characterize all bounded Toeplitz operators which commute with $T_{e^{ip\theta}r^m}$, where m and p are integers.

A direct calculation gives the following lemma which we shall use often.

Lemma 1. *Let $k, p \in \mathbb{N}$ and let f be an integrable radial function. Then*

$$T_{e^{ip\theta}f}(z^k) = 2(k+p+1)\widehat{f}(2k+p+2)z^{k+p}$$

and

$$T_{e^{-ip\theta}f}(z^k) = \begin{cases} 0 & \text{if } 0 \leq k \leq p-1 \\ 2(k-p+1)\widehat{f}(2k-p+2)z^{k-p} & \text{if } k \geq p\,. \end{cases}$$

Proof. For $p \geq 0$ and all $k \in \mathbb{N}$, we have

$$\begin{aligned} T_{e^{ip\theta}f}(z^k) = P(e^{ip\theta}fz^k) &= \sum_{n\geq 0}(n+1)\langle e^{ip\theta}fz^k, z^n\rangle z^n \\ &= \sum_{n\geq 0}(n+1)\left(\int_0^1\int_0^{2\pi} f(r)r^{k+n+1}e^{k+p-n}\frac{d\theta}{\pi}dr\right)z^n \\ &= 2(k+p+1)\widehat{f}(2k+p+2)z^{k+p}\,. \end{aligned}$$

A similar calculation gives the values of the quasihomogeneous Toeplitz operator of negative degree on the elements of the basis of L^2_a. $\square$

3. Product of n Toeplitz operators

It was shown in [1], that if $T_f T_g = 0$ with both f and g harmonic, then one of the symbols must be equal to the zero function. This result was extended in [10], by showing that the same result remains true if we assume that the product $T_f T_g$ is of finite rank. In [2], it was proved that if one of the symbols f or g is a radial function and the other is arbitrary and if $T_f T_g = 0$, then also f or g should be zero. However the "zero product" problem without any restriction on the symbols is still open.

In our first theorem, we show that, if the product of n quasihomogeneous Toeplitz operators is of finite rank, then one of the symbols must be zero.

Theorem 2. *Let $p_1, p_2, \dots, p_m$ be integers and let $f_1, \dots, f_m$ be bounded radial functions on $\mathbb{D}$. If the product $T_{e^{ip_m\theta} f_m} \dots T_{e^{ip_1\theta} f_1}$ is of finite rank N, then $f_j = 0$ for some $j \in \{1, \dots, m\}$.*

Proof. We denote by S the product of Toeplitz operators $T_{e^{ip_m\theta} f_m} \dots T_{e^{ip_1\theta} f_1}$. For every $k \geq \sum_{j=1}^m |p_j|$, we have

$$\begin{aligned} S(z^k) &= 2(k + p_1 + 1)\widehat{f_1}(2k + p_1 + 2)2(k + p_1 + p_2 + 1)\widehat{f_2}(2k + 2p_1 + p_2 + 2) \\ &\quad \dots 2(k + p_1 + \dots + p_m + 1)\widehat{f_m}(2k + 2p_1 + \dots + p_m + 2)z^{k+p_1+\dots+p_m} . \end{aligned}$$

Thus the set $\{S(z^k) : k \geq \sum_{j=1}^m |p_j|\}$ is a linearly independent set which is included in the range of S. Hence $\{S(z^k) : k \geq \sum_{j=1}^m |p_j|\}$ contains at most N elements. This implies that there exists some positive integer $n_0 \geq N + \sum_{j=1}^m |p_j|$ such that

$$S(z^k) = 0 \quad \text{for all} \quad k \geq n_0 ,$$

which is equivalent to

$$\widehat{f_1}(2k+p_1+2) \dots \widehat{f_m}(2k+2p_1+\dots+2p_{m-1}+p_m+2) = 0 , \quad \text{for all} \quad k \geq n_0 . \quad (1)$$

Let $l = \min\{p_1, 2p_1 + p_2, \dots, 2p_1 + \dots + 2p_{m-1} + p_m\}$. Then (1) implies that

$$(r^{p_1 - l} f_1)^\wedge(2k + l + 2) \dots (r^{2p_1+\dots+p_m-l} f_m)^\wedge(2k + l + 2) = 0 , \quad \text{for all} \quad k \geq n_0 .$$

Thus the function $(r^{p_1-l} f_1)^\wedge \dots (r^{2p_1+\dots+p_m-l} f_m)^\wedge$, which is the product of bounded analytic functions in the right half-plane $\{z : \Re z > 2\}$, vanishes on the arithmetic sequence $(2k + l + 2)_{k \geq n_0}$. By Theorem 1, it must be zero and hence at least one of its bounded analytic factors should be equal to zero, i.e., there exists some $j \in \{1, \dots m\}$ such that $f_j = 0$. □

4. Finite rank semicommutators

On the Hardy space H^2 of the unit disk $\mathbb{D}$, the question for which symbols f and g the semicommutator $(T_f, T_g]$ or the commutator $[T_f, T_g]$ is of finite rank, has been completely solved in [3] and [9]. On the Bergman space, the zero semicommutator or commutator of two Toeplitz operators with harmonic symbols has been completely characterized in [4] and [18]. Recently, those results were generalized by

Guo, Sun and Zheng [10]. In fact they proved that if the semicommutator or the commutator of two Toeplitz operators with bounded harmonic symbols has finite rank, then it must be zero.

The situation for the semicommutator involving quasihomogeneous Toeplitz operators is different. We shall show that, for p and s both positive, if the semicommutator $(T_{e^{ip\theta}f}, T_{e^{is\theta}g}]$ has finite rank, then the semicommutator must be zero. However if p and s have opposite signs, then we prove that there exists a nonzero semicommutator $(T_{e^{ip\theta}f}, T_{e^{is\theta}g}]$ of finite rank. The main ingredient in our proofs is Theorem 6.1 in [11], which gives necessary and sufficient conditions for the product of two quasihomogeneous Toeplitz operators to be a Toeplitz operator too. Also in certain cases, it gives an explicit formula for the symbol of the product. This formula is essentially a Mellin convolution equation. We recall that the Mellin convolution of two radial functions ϕ and ψ in $L^1([0,1], rdr)$, denoted by $\phi *_M \psi$, is defined by:

$$(\phi *_M \psi)(r) = \int_r^1 \phi\left(\frac{r}{t}\right) \psi(t) \frac{dt}{t}, \quad \text{for} \quad 0 \le r < 1 .$$

It is easy to see that the Mellin transform converts a convolution product into a pointwise product, i.e., that

$$\widehat{(\phi *_M \psi)}(r) = \widehat{\phi}(r)\widehat{\psi}(r)$$

and that, if ϕ and ψ are in $L^1([0,1], rdr)$ then so is $\phi *_M \psi$.

Below we state Theorem 6.1 in [11] for completeness.

Theorem 3. *Let $p, s \in \mathbb{N}$, $p \ge s$ and let f and g be two integrable radial functions on $\mathbb{D}$ such that $T_{e^{ip\theta}f}$ and $T_{e^{-is\theta}g}$ are bounded. Then*

$$T_{e^{ip\theta}f} T_{e^{-is\theta}g}$$

is equal to a Toeplitz operator if and only if there exists an integrable radial function h such that

(a) *$T_{e^{i(p-s)\theta}h}$ is bounded;*
(b) *$\widehat{h}(2k + p - s + 2) = 0$ if $0 \le k \le s - 1$;*
(c) *h is a solution to the equation*

$$\mathbb{1} *_M r^{p+s} h = r^p f *_M r^s g ,$$

where $\mathbb{1}$ denotes the constant function with value one.

In this case:

$$T_{e^{ip\theta}f} T_{e^{-is\theta}g} = T_{e^{i(p-s)\theta}h} .$$

Remark 2. The case where both Toeplitz operators are of positive quasihomogeneous degrees can be treated similarly. Under the same hypothesis as in the theorem above, the product $T_{e^{ip\theta}f} T_{e^{is\theta}g}$ is a Toeplitz operator if and only if there exists an integrable radial function h such that

(a) $T_{e^{i(p+s)\theta}h}$ is bounded;

(b) h is a solution to the equation

$$r^s *_M r^p h = r^{p+s} f *_M g\,.$$

In this case:

$$T_{e^{ip\theta}f} T_{e^{is\theta}g} = T_{e^{i(p+s)\theta}h}\,.$$

We are now ready to state and prove our results about semicommutators.

Theorem 4. *Let p and s be two positive integers and let f and g be two integrable radial functions on $\mathbb{D}$ such that $T_{e^{ip\theta}f}$, $T_{e^{is\theta}g}$ and $T_{e^{i(p+s)\theta}fg}$ are bounded operators. If the semicommutator $(T_{e^{ip\theta}f}, T_{e^{is\theta}g}]$ has finite rank, then it is equal to zero.*

Proof. Let denote by S the semicommutator $(T_{e^{ip\theta}f}, T_{e^{is\theta}g}]$. Suppose that S has a finite rank N. As in the proof of the Theorem 2, the range of S contains at most N linearly independent vectors. Thus there exists $n_0 \geq N$, such that

$$S(z^k) = 0 \quad \text{for all} \quad k \geq n_0\,.$$

Lemma 1 implies that

$$2(k+s+1)\widehat{f}(2k+p+2s+1)\widehat{g}(2k+s+2) = \widehat{fg}(2k+p+s+2)\,, \tag{2}$$

for all $k \geq n_0$. Since:

$$\widehat{\mathbb{1}}(2k+2s+2) = \widehat{r^s}(2k+s+2) = \frac{1}{2(k+s+1)}\,,$$

then (2) is equivalent to:

$$(\widehat{r^{p+s} f *_M g})(2k+s+2) = (\widehat{r^s *_M r^p} fg)(2k+s+2)\,.$$

Now the functions $(\widehat{r^{p+s} f *_M} g)$ and $(r^s \widehat{*_M r^p} fg)$ are both analytic on the right half-plane $\{z : \Re z > 2\}$ and the sequence $(2k+s+2)_{k \geq n_0}$ is arithmetic, then Remark 1 implies that

$$r^{p+s} f *_M g = r^s *_M r^p fg\,.$$

Hence $T_{e^{ip\theta}f} T_{e^{is\theta}g} = T_{e^{i(p+s)\theta}fg}$ by Remark 2. □

Remark 3. If p and s are both negative, the same theorem above remains true by considering the adjoint of the semicommutator.

We shall next consider the case of a semicommutator of two quasihomogeneous Toeplitz whose degrees have opposite signs.

Theorem 5. *Let $p \geq s$ be two positive integers and let f and g be two integrable radial functions on $\mathbb{D}$ such that $T_{e^{ip\theta}f}$, $T_{e^{is\theta}g}$ and $T_{e^{i(p-s)\theta}fg}$ are bounded operators. If the semicommutator $(T_{e^{ip\theta}f}, T_{e^{-is\theta}g}]$ has finite rank N, then N is at most equal to the quasihomogeneous degree s.*

Proof. Let denote by S the semicommutator $(T_{e^{ip\theta}f}, T_{e^{-is\theta}g}]$. By Lemma 1, we have

$$S(z^k) = \begin{cases} 2(k+p-s+1)\widehat{fg}(2k+p-s+2)z^{k+p-s}\,, & \text{if } k \le s-1 \\ 2(k+p-s+1)\big[\widehat{fg}(2k+p-s+2)- & \\ 2(k-s+1)\widehat{f}(2k+p-2s+2)\widehat{g}(2k-s+2)\big]z^{k+p-s}\,, & \text{if } k \ge s\,. \end{cases}$$

If S has finite rank N, then as before there exists $n \ge s$ such that

$$S(z^k) = 0 \quad \text{for} \quad k \ge n\,.$$

The preceding equation is equivalent to

$$\widehat{fg}(2k+p-s+2) = 2(k-s+1)\widehat{f}(2k+p-2s+2)\widehat{g}(2k-s+2)\,, \quad \text{for} \quad k \ge n\,.$$

Using the fact that $\widehat{\mathbb{1}}(2k-2s+2) = \frac{1}{2(k-s+1)}$, we obtain

$$\widehat{\mathbb{1}}(2k-2s+2)\widehat{r^{p+s}fg}(2k-2s+2) = \widehat{r^p f}(2k-2s+2)\widehat{r^s g}(2k-2s+2)\,, \quad \text{for} \quad k \ge n\,. \tag{3}$$

Since the sequence $(2k-2s+2)_{k\ge n}$ is arithmetic, then Remark 1 and (3) imply that

$$\widehat{\mathbb{1}}(z)\widehat{r^{p+s}fg}(z) = \widehat{r^p f}(z)\widehat{r^s g}(z)\,, \quad \text{for} \quad \Re z > 0\,.$$

In particular if $z = 2k-2s+2$ with $k \ge s$, we have

$$\widehat{\mathbb{1}}(2k-2s+2)\widehat{r^{p+s}fg}(2k-2s+2) = \widehat{r^p f}(2k-2s+2)\widehat{r^s g}(2k-2s+2)\,, \quad \text{for} \quad k \ge s\,.$$

Hence

$$S(z^k) = 0 \quad \text{for} \quad k \ge s\,.$$

Therefore the rank N of S is at most equal to s. □

Below we give an effective construction of a nonzero semicommutator of finite rank.

Example 1. Using Theorem 3, one can find many nonzero finite rank semicommutators of quasihomogeneous Toeplitz operators. Using the notation of Theorem 3, let $p = s = 1$ and let $f(r) = 1/r$. The function f is not bounded but it is a so called "nearly bounded function" [2, p. 204]. Thus the Toeplitz operator with symbol $e^{i\theta}1/r$ is bounded. Now it is easy to see that any radial function g will satisfy the Mellin convolution equation in the condition (c) of Theorem 3 with $h = fg$. Take for example $g(r) = r$. Using Lemma 1, we have that

$$T_{e^{i\theta}\frac{1}{r}}T_{e^{-i\theta}r}(z^k) = T_{\mathbf{1}}(z^k)\,, \quad \text{for} \quad k \ge 1\,.$$

However

$$T_{e^{i\theta}\frac{1}{r}}T_{e^{-i\theta}r}(1) = 0\,,$$

but

$$T_{\mathbf{1}}(1) = 1\,.$$

Therefore $(T_{e^{i\theta}\frac{1}{r}}, T_{e^{-i\theta}r}]$ has rank one.

5. Finite rank commutators

We now pass to the commutator of two quasihomogeneous Toeplitz operators. Here the situation is the same as for the semicommutator. We shall prove that if two quasihomogeneous Toeplitz operators have both positive or both negative degrees and if their commutator has finite rank, then the commutator must be zero. However if the degrees of operators have opposite signs, then the commutator can have finite rank without being equal to zero.

Theorem 6. *Let p and s be two positive integers and let f and g be two integrable radial functions on $\mathbb{D}$ such that $T_{e^{ip\theta}f}$ and $T_{e^{is\theta}g}$ are bounded operators. If the commutator $[T_{e^{ip\theta}f}, T_{e^{is\theta}g}]$ has a finite rank, then it is equal to zero.*

Proof. Let S denote the commutator $[T_{e^{ip\theta}f}, T_{e^{is\theta}g}]$. If S has finite rank N, then as before there exists $n \geq N$ such that

$$S(z^k) = 0\,, \quad \text{for} \quad k \geq n\,,$$

which is equivalent, using Lemma 1, to

$$2(k+s+1)\widehat{f}(2k+p+2s+2)\widehat{g}(2k+s+2) = 2(k+p+1)\widehat{f}(2k+p+2)\widehat{g}(2k+2p+s+2)\,,$$

for $k \geq n$. Since $\widehat{\mathbb{1}}(2k+2s+2) = \frac{1}{2(k+s+1)}$ and $\widehat{\mathbb{1}}(2k+2p+2) = \frac{1}{2(k+p+1)}$, then the preceding equality implies that

$$\widehat{\mathbb{1}}(2k+2p+2)\widehat{f}(2k+p+2s+2)\widehat{g}(2k+s+2) = \widehat{\mathbb{1}}(2k+2s+2)\widehat{f}(2k+p+2)\widehat{g}(2k+2p+s+2)\,,$$

for all $k \geq n$. Now each of the functions $\widehat{\mathbb{1}}$, $\widehat{f}$ and $\widehat{g}$ is bounded analytic function in the right half-plane $\{z : \Re z > 2\}$ and the sequence $(2k+2)_{k\geq n}$ is arithmetic, then using Remark 1, we have that

$$\widehat{\mathbb{1}}(z+2p)\widehat{f}(z+p+2s)\widehat{g}(z+s) = \widehat{\mathbb{1}}(z+2s)\widehat{f}(z+p)\widehat{g}(z+2p+s)\,, \quad \text{for} \quad \Re z \geq 2\,.$$

In particular for $z = 2k+2$, with $k \geq 0$, we obtain that

$$\widehat{\mathbb{1}}(2k+2p+2)\widehat{f}(2k+p+2s+2)\widehat{g}(2k+s+2) = \widehat{\mathbb{1}}(2k+2s+2)\widehat{f}(2k+p+2)\widehat{g}(2k+2p+s+2)\,,$$

and this is true for all $k \geq 0$. Hence

$$S(z^k) = 0\,, \quad \text{for} \quad k \geq 0\,.$$

Therefore the commutator $[T_{e^{ip\theta}f}, T_{e^{is\theta}g}]$ equals to zero. □

Theorem 7. *Let $p \geq s$ be two positive integers and let f and g be two integrable radial functions such that the Toeplitz operators $T_{e^{ip\theta}f}$ and $T_{e^{-is\theta}g}$ are bounded. If the commutator $[T_{e^{ip\theta}f}, T_{e^{-is\theta}g}]$ has finite rank N, then N is at most equal to the quasihomogeneous degree s.*

Proof. Let S denotes the commutator $[T_{e^{ip\theta}f}, T_{e^{-is\theta}g}]$. Then
if $k \leq s-1$:

$$S(z^k) = -2(k+p-s+1)\widehat{g}(2k+2p-s+2)2(k+p+1)\widehat{f}(2k+p+2)z^{k+p-s}\,,$$

and if $k \geq s$:

$$S(z^k) = 2(k+p-s+1)\big[2(k-s+1)\widehat{f}(2k+p-2s+2)\widehat{g}(2k-s+2) \\ - 2(k+p+1)\widehat{g}(2k+2p-s+2)\widehat{f}(2k+p+2)\big]z^{k+p-s}\,.$$

If S has finite rank N, then there exists $n \geq s$ such that

$$S(z^k) = 0\,, \quad \text{for all} \quad k \geq n\,.$$

Hence, for every $k \geq n$, we have

$$2(k-s+1)\widehat{f}(2k+p-2s+2)\widehat{g}(2k-s+2) = 2(k+p+1)\widehat{g}(2k+2p-s+2)\widehat{f}(2k+p+2)\,.$$

As in the proof of Theorem 5, the above equation implies that

$$2(k-s+1)\widehat{f}(2k+p-2s+2)\widehat{g}(2k-s+2) = 2(k+p+1)\widehat{g}(2k+2p-s+2)\widehat{f}(2k+p+2)\,,$$

for all $k \geq s$. Thus

$$S(z^k) = 0\,, \quad \text{for} \quad k \geq s\,.$$

Therefore the rank N of S is less or equal to s. □

Here we present an example of a nonzero commutator of finite rank.

Example 2. It has been proved in [12, Proposition 8, p. 252], that quasihomogeneous Toeplitz operators whose quasihomogeneous degrees have opposite signs never commute. Below we will construct a radial function g such that

$$T_{e^{ip\theta}r^m}T_{e^{-is\theta}g}(z^k) = T_{e^{-is\theta}g}T_{e^{ip\theta}r^m}(z^k)\,, \quad \text{for} \quad k \geq s\,, \tag{4}$$

where $p \geq s > 0$ and m are given positive integers.

Equation (4) implies that for $k \geq s$, we have

$$\frac{2k-2s+2}{2k+p-2s+m+2}\widehat{g}(2k-s+2) = \frac{2k+2p+2}{2k+p+m+2}\widehat{g}(2k+2p-s+2)\,.$$

Thus for $k \geq s$

$$\frac{\widehat{r^{-s}g}(2k+2p+2)}{\widehat{r^{-s}g}(2k+2)} = \frac{(2k-2s+2)(2k+p+m+2)}{(2k+2p+2)(2k+p-2s+m+2)}\,.$$

Now, using Remark 1, we obtain that

$$\frac{\widehat{r^{-s}g}(z+2p)}{\widehat{r^{-s}g}(z)} = \frac{(z-2s)(z+p+m)}{(z+2p)(z+p-2s+m)}\,, \quad \text{for} \quad \Re z \geq s+2\,. \tag{5}$$

Let F be the analytic function defined for $\Re z > 2s$ by

$$F(z) = \frac{\Gamma\left(\frac{z-2s}{2p}\right)\Gamma\left(\frac{z+p+m}{2p}\right)}{\Gamma\left(\frac{z+2p}{2p}\right)\Gamma\left(\frac{z+p-2s+m}{2p}\right)}\,,$$

where Γ denotes the gamma function. Then, using the well-know identity $\Gamma(z+1) = z\Gamma(z)$, (5) implies that

$$\frac{\widehat{r^{-s}g}(z+2p)}{\widehat{r^{-s}g}(z)} = \frac{F(z+2p)}{F(z)}, \quad \text{for} \quad \Re z > 2s\,. \tag{6}$$

Equation (6), combined with [13, Lemma 6, p. 1428], gives us that there exists a constant c such that

$$\widehat{r^{-s}g}(z) = cF(z)\,, \quad \text{for} \quad \Re z > 2s\,. \tag{7}$$

For a choice of $p = 2$, $s = 1$ and $m = 6$, and again using the identity $\Gamma(z+1) = z\Gamma(z)$, one can see that

$$F(z) = \frac{z+4}{(z-2)(z+2)} = \frac{3}{2}\frac{1}{z-2} - \frac{1}{2}\frac{1}{z+2}\,.$$

Since $\frac{1}{z\pm 2} = \widehat{r^{\pm 2}}(z)$, then (7) becomes

$$\widehat{r^{-1}g}(z) = \frac{c}{2}\big(3\widehat{r^{-2}}(z) - \widehat{r^{2}}(z)\big)\,, \quad \text{for} \quad \Re z > 2\,.$$

Now the preceding equation and Remark 1 imply that

$$g(r) = c\left(\frac{3}{r} - r^3\right), \quad \text{where } c \text{ is a constant}\,.$$

It is clear that the function g is not bounded but it is nearly bounded so that the Toeplitz operator $T_{e^{-i\theta}g}$ is bounded.

Finally, by taking the constant c to be equal to 1, the radial function $g(r) = 3/r - r^3$ satisfies

$$T_{e^{2i\theta}r^6}T_{e^{-i\theta}g}(z^k) = T_{e^{-i\theta}g}T_{e^{2i\theta}r^6}(z^k)\,, \quad \text{for} \quad k \geq 1\,.$$

However, using Lemma 1, it is easy to see that

$$T_{e^{2i\theta}r^6}T_{e^{-i\theta}g}(1) = 0\,,$$

but

$$T_{e^{-i\theta}g}T_{e^{2i\theta}r^6}(1) \neq 0\,.$$

Therefore the commutator $[T_{e^{2i\theta}r^6}, T_{e^{-i\theta}(\frac{3}{r}-r^3)}]$ has rank one.

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