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Bicommutants of Toeplitz operators

I. Louhichi and N. V. Rao

Abstract. In this paper we discuss an unusual phenomenon in the context of Toeplitz operators in the Bergman space on the unit disc: If two Toeplitz operators commute with a quasihomogeneous Toeplitz operator, then they commute with each other. In the Bourbaki terminology, this result can be stated as follows: The commutant of a quasihomogeneous Toeplitz operator is equal to its bicommutant.

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Let $\mathbb{D}$ denote the unit disc in the complex plane $\mathbb{C}$, and $dA = rdr\frac{d\theta}{\pi}$, where (r, θ) are polar coordinates, is the normalized Lebesgue measure. So that $\mathbb{D}$ has area 1. The Bergman space L^2_a is the Hilbert space of analytic functions on $\mathbb{D}$ that are square integrable with respect to the measure dA. It is well known that L^2_a is a closed subspace of the Hilbert space $L^2(\mathbb{D}, dA)$ and has $\{\sqrt{n+1}z^n \mid n \geq 0\}$ as an orthonormal basis. Let P be the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto L^2_a. For any function $\phi \in L^\infty(\mathbb{D}, dA)$, the Toeplitz operator T_ϕ with symbol ϕ is the operator on L^2_a defined by $T_\phi f = P(\phi f)$, for any $f \in L^2_a$.

The question when two Toeplitz operators in the Bergman space commute, was worked on by many people [1, 4, 5, 2, 6, 7, 9] since Brown and Halmos solved the analogous problem on the Hardy space of the unit circle $\mathbb{T}$ of $\mathbb{C}$ [3]. In fact they prove that $T_\phi T_\psi = T_\psi T_\phi$ for some ϕ and ψ in $L^\infty(\mathbb{T})$ if and only if

(a) ϕ and ψ are both analytic,
or
(b) $\bar{\phi}$ and $\bar{\psi}$ are both analytic,
or
(c) one of the two symbols is a linear function of the other.

The main motivation for this paper is to show a rare phenomenon, in the context of Toeplitz operators in the Bergman space, which can be stated as follows:

There exists a large class of Toeplitz operators T such that if two other Toeplitz operators S and U commute with T, then S commutes with U. In the Bourbaki terminology, our result can be stated as follows: The commutant of T is equal to its bicommutant in the class $\mathcal{T}$ of all Toeplitz operators in the Bergman space. [*The commutant of T is the set of all those operators that commute with it and bicommutant is the set of all operators that commute with all operators in the commutant.*]

This was noticed by the second author in the case of operators with symbols $e^{ip\theta}r^m$ [8]. About the same time the first author [7] found that the quasihomogeneous operators, which he and Zakariasy studied in [6], that commute with operators with symbol $e^{ip\theta}r^m$ tend to be powers of a single operator if $\frac{m}{p}$ is an odd positive integer. Thus the bicommutant of any such quasihomogeneous Toeplitz operator T in $\mathcal{T}$ is a maximal commutative algebra contained in $\mathcal{T}$ and generated by T. Proof of this will be furnished in this paper.

Now we will introduce some notation and also quote some results of [6, 7] to facilitate the presentation of our results. We also introduce a new class of operators called *Holomorphic Weighted Shift* operators, HWS for short, which improves our presentation enormously and indeed a lot of the previous work in [5, 6, 7] if presented in this form can be understood with great ease though we shall not do the clean up on this occasion.

In [6], Louhichi and Zakariasy extended the results in [5] by replacing $e^{ip\theta}r^m$ with a symbol of type $e^{ip\theta}\phi(r)$ where ϕ is an arbitrary radial function and p is an integer. This type of symbol is called a **quasihomogeneous** function of order p and the associated Toeplitz operator is also called quasihomogeneous Topelitz operator of order p. The reason to study such a family of symbols is that any function f in $L^2(\mathbb{D}, dA)$ has the following polar decomposition

$$L^2(\mathbb{D}, dA) = \bigoplus_{k\in\mathbb{Z}} e^{ik\theta}\mathcal{R}$$

where $\mathcal{R}$ is the space of functions on $[0,1]$ that are square integrable with respect to the measure rdr. The study of quasihomogeneous Toeplitz operators allows us to obtain interesting results about Toeplitz operators with more general symbols. We need the results from [6] for our purposes and they can be summarized as follows:

Theorem LZ. *Let ϕ be a nonzero bounded radial function, p be a positive integer and $f(re^{ik\theta}) = \sum_{k=-\infty}^{+\infty} e^{ik\theta} f_k(r) \in L^\infty(\mathbb{D}, dA)$. Then*

a) *T_f commutes with $T_{e^{ip\theta}\phi}$ if and only if $T_{e^{ik\theta}f_k}$ commutes with $T_{e^{ip\theta}\phi}$ for all $k \in \mathbb{Z}$.*

b) *If there exists $k \in \mathbb{Z}_-$ and a bounded radial function f_k such that*

$$T_{e^{ip\theta}\phi} T_{e^{ik\theta} f_k} = T_{e^{ik\theta} f_k} T_{e^{ip\theta}\phi}$$

then f_k must be equal to zero.

c) *If there exists $k \in \mathbb{Z}_+$ and a bounded radial function f_k such that*

$$T_{e^{ip\theta}\phi} T_{e^{ik\theta} f_k} = T_{e^{ik\theta} f_k} T_{e^{ip\theta}\phi}$$

then f_k is unique up to a constant factor. In particular f_0 is a constant.

Remark 1. The above theorem is true for $p < 0$ also, which can be seen by taking the adjoints. For this reason we deal only with $p > 0$.

In [7] the first author obtained a relationship between commutativity, roots and powers of the quasihomogeneous Toeplitz operators. We borrow proposition 7 of this paper and its techniques to generalize for our case. For completeness, we chose to state it here as follows:

Theorem L. *Let p and s be two positive integers and ϕ and ψ be two nonzero bounded radial functions such that*

$$T_{e^{ip\theta}\phi} T_{e^{is\theta}\psi} = T_{e^{is\theta}\psi} T_{e^{ip\theta}\phi}.$$

Then

$$\Big(T_{e^{ip\theta}\phi}\Big)^s = c\Big(T_{e^{is\theta}\psi}\Big)^p,$$

for some constant c.

The Mellin transform of a function $\phi \in L^1([0,1], rdr)$ is defined by

$$\widehat{\phi}(z) = \int_0^1 \phi(r) r^{z-1}\, dr$$

It is easy to see that $\widehat{\phi}$ is a bounded holomorphic function on the half-plane $\{z \in \mathbb{C} | \operatorname{Re} z > 2\}$.
Let $\phi \in L^\infty(\mathbb{D}, dA)$ be a radial function and let p be a positive integer. Then

$$T_{e^{ip\theta}\phi}(\zeta^k)(z) = \int_0^1 \int_0^{2\pi} \phi(r) r^k \sum_{j=0}^{\infty} (j+1) e^{i(k+p-j)\theta} r^j z^j \frac{1}{\pi}\, r dr d\theta$$

and interchanging the integral over $[0, 2\pi]$ and the sum we see that

$$T_{e^{ip\theta}\phi}(\zeta^k)(z) = 2(k+p+1)\widehat{\phi}(2k+p+2) z^{k+p}.$$

Thus $T_{e^{ip\theta}\phi}$ acts on the elements of the orthogonal basis of L^2_a as a shift operator with a holomorphic weight. In fact if we denote by

$$F(z) = 2(z+p+1)\widehat{\phi}(2z+p+2),$$

then $T_{e^{ip\theta}\phi}(\zeta^k)(z) = F(k)z^{k+p}$. It is clear that F is a holomorphic function in the right-half plane $\{z \in \mathbb{C} | \operatorname{Re} z > -\frac{p}{2}\}$. To exploit well this observation, we introduce the following definition.

Definition 1. Let F be a holomorphic function in the right-half plane $\{z \in \mathbb{C} | \operatorname{Re} z \geq 0\}$, we define the Holomorphic Weighted Shift, denoted HWS, operator T_F of symbol F and order p on L_a^2 by

$$T_F(z^k) = F(k)z^{k+p}.$$

Proposition 1. *T_F is bounded if and only if F is bounded on $\mathbb{N}_0$, the set of all nonnegative integers.*

We include a short proof of this for completeness.

Proof. If T_F is a bounded HWS of order p on L_a^2, then for all $k \geq 0$ we have

$$\|T_F(z^k)\|_2 = |F(k)| \|z^{k+p}\|_2 \leq \|T_F\| \|z^k\|_2. \tag{1}$$

Since $\|z^k\|_2 = \sqrt{\dfrac{2}{2k+2}}$, hence (1) implies that for all $k \geq 0$

$$|F(k)| \sqrt{\frac{2}{2k+2p+2}} \leq \|T_F\| \sqrt{\frac{2}{2k+2}},$$

which is equivalent to

$$|F(k)| \leq \|T_F\| \sqrt{\frac{2k+2p+2}{2k+2}}, \text{ for all } k \geq 0.$$

It is clear that the quantity $\sqrt{\dfrac{2k+2p+2}{2k+2}}$ is bounded independently from k, hence F is bounded.
Conversely, suppose that F is bounded on $\mathbb{N}_0$ and $\|F\|_\infty = \sup_{\mathbb{N}_0} |F(k)|$. Each function f in the Bergman space L_a^2 has a power series expansion

$$f(z) = \sum_{k=0}^{\infty} a_k z^k.$$

Let $f_n = \sum_{k=0}^{n} a_k z^k$. Now

$$\begin{aligned}
\|T_F f_n\|_2 = \left\| T_F(\sum_{k=0}^{n} a_k z^k) \right\|_2 &= \left\| \sum_{k=0}^{n} a_k F(k) z^{k+p} \right\|_2 \\
&= \sqrt{\sum_{k=0}^{n} |a_k|^2 |F(k)|^2 \|z^{k+p}\|_2^2} \\
&= \sqrt{\sum_{k=0}^{n} |a_k|^2 |F(k)|^2 \left(\frac{2}{2k+2p+2} \right)} \\
&= \|F\|_\infty \sqrt{\sum_{k=0}^{n} \frac{2|a_k|^2}{2k+2p+2}} \\
&\le \|F\|_\infty \sqrt{\sum_{k=0}^{n} \frac{2|a_k|^2}{2k+2}} \\
&= \|F\|_\infty \|f_n\|_2.
\end{aligned}$$

Letting $n \to \infty$ we see that any holomorphic function F bounded on $\mathbb{N}_0$ defines a HWS operator T_F. □

Remark 2. (a) A bounded quasihomogeneous Toeplitz operator is a HWS operator.
(b) The product of two HWS operators of order respectively p and q is a HWS operator of order $p+q$.

We shall often use the following classical lemma.

Lemma 1. *If a meromorphic function in a right half plane belonging to the Nevanlinna class is periodic, then it is constant.*

The following proposition generalizes the Theorem L and its proof.

Proposition 2. *Let T_F and T_G be two HWS operators of order respectively p and q both positive integers and also we assume that F, G are holomorphic in the closure of right half plane and bounded. If T_F commutes with T_G then*

$$T_F^q = c T_G^p,$$

for some constant c.

Proof. By Remark 2, T_F^q is a HWS operator of order pq and its symbol, denoted A, is given by

$$A(k) = F(k)F(k+p)\dots F(k+(q-1)p).$$

Also T_G^p is a HWS operator of order pq and its symbol, denoted B, is given by

$$B(k) = G(k)G(k+q)\dots G(k+(p-1)q).$$

Thus $T_F^q = T_A$ and $T_G^p = T_B$. Since by hypothesis T_F and T_G commute then T_A and T_B will also commute, which means that for all $k \geq 0$

$$T_A T_B(z^k) = T_B T_A(z^k),$$

and hence, for all $k \geq 0$,

$$A(k+pq)B(k) = A(k)B(k+pq).$$

Now Lemma 1 implies that for all $k \geq 0$, $A(k) = cB(k)$, where c is a constant. The proof is complete. $\square$

Remark 3. The same argument of the above proof can be used to show that in fact if T_F and T_G commute, $T_F^n = cT_G^m$ for some constant c and any two positive integers m, n such that $np = mq$. This improves the Theorem L.

The two lemmas below will allow us to prove that if two quasihomogeneous Toeplitz operators $T_{e^{ip\theta}\phi}$ and $T_{e^{is\theta}\psi}$ of positive quasihomogeneous degrees p and s respectively, are such that $\left(T_{e^{ip\theta}\phi}\right)^m = \left(T_{e^{is\theta}\psi}\right)^n$ for a pair (m,n) of positive integers, then $T_{e^{ip\theta}\phi}$ and $T_{e^{is\theta}\psi}$ commute.

Lemma 2. *Let T and S be two HWS operators of order p. If there exist a positive integer d such that $T^d = S^d$, then $T = cS$ where c is a d^{th} root of unity.*

Proof. Since T and S are of the same order p, then T^d and S^d are both HWS operators of order pd. By hypothesis, $T^d(z^k) = S^d(z^k)$ for all $k \geq 0$, which implies that

$$T(z)T(z+p)\dots T(z+dp) = S(z)S(z+p)\dots S(z+dp).$$

If we multiply the above equation by the equation obtained by replacing z by $z+p$, we have

$$T(z)S(z+p+dp) = S(z)T(z+p+dp). \tag{2}$$

Now, equation (2) implies that the meromorphic function $\dfrac{T}{S}$ is periodic of periodicity $(d+1)p$. Hence Lemma 1 implies that $T = cS$ where c is a constant. $\square$

Lemma 3. *Let T_F be a HWS operator of symbol F and order p and let T_G be a HWS operator of symbol G and order q. If there exist two coprime integers n and m such that $T_F^m = T_G^n$, then $T_F T_G = T_G T_F$.*

Proof. Since $T_F^m = T_G^n$, then $T_F T_G^n = T_G^n T_F$. Thus we have

$$F(z+nq)\prod_{j=0}^{n-1} G(z+jq) = F(z)\prod_{j=0}^{n-1} G(z+p+jq).$$

Hence

$$\prod_{j=0}^{n-1} F(z+q+jq)G(z+jq) = \prod_{j=0}^{n-1} F(z+jq)G(z+p+jq). \tag{3}$$

If we denote by $A(z) = F(z+q)G(z)$ and $B(z) = F(z)G(z+p)$, then the equation (3) becomes

$$A(z)A(z+q)\ldots A(z+(n-1)q) = B(z)B(z+q)\ldots B(z+(n-1)q). \tag{4}$$

Replacing z by $z+q$ in the equation (4), we obtain

$$A(z+q)A(z+2q)\ldots A(z+nq) = B(z+q)B(z+2q)\ldots B(z+nq). \tag{5}$$

From equations (4) and (5), we have

$$A(z)B(z+nq) = A(z+nq)B(z).$$

Now Lemma 1 implies that there exists a constant c such that $A(z) = cB(z)$ and $c^n = 1$. Thus

$$F(z+q)G(z) = cF(z)G(z+p),$$

which is equivalent to say that $T_F T_G = cT_G T_F$. Redoing the same argument with roles of F and G reversed implies that there exists a constant d such that $T_G T_F = dT_F T_G$ and $d^m = 1$. But $d = \frac{1}{c}$, so $c^m = 1$ and hence $c = 1$ because m and n are coprime. □

Proposition 3. *Let T_F and T_G be two HWS operators. Suppose that there exist two positive integers m and n such that $T_F^m = T_G^n$. Then $T_F T_G = T_G T_F$.*

Proof. Let l be the greatest common divisor of m and n. Then by hypothesis

$$\left(T_F^{\frac{m}{l}}\right)^l = \left(T_G^{\frac{n}{l}}\right)^l \text{ with } \frac{m}{l} \text{ and } \frac{n}{l} \text{ are coprime.}$$

Since $T_F^{\frac{m}{l}}$ and $T_G^{\frac{n}{l}}$ are both HWS operators of the same order, then Lemma 2 implies that there exists a constant c_0 such that $T_F^{\frac{m}{l}} = c_0 T_G^{\frac{n}{l}}$ and $c_0^l = 1$. Using the linearity of the Toeplitz operator about its symbol, one can write that $c_0 T_G^{\frac{n}{l}} = \left(T_{c_1 G}\right)^{\frac{n}{l}}$ with $c_1^{\frac{l}{n}} = c_0$. Now by Lemma 3, we have that T_F and $T_{c_1 G}$ commute which is equivalent to say that T_F and T_G commute. □

Now we are ready to state our main result.

Theorem 1. *Let T_F, T_G and T_H be HWS operators of order p, q and s respectively. Suppose that T_G and T_H commute with T_F. Then T_G and T_H commute with each other.*

Proof. Since T_G and T_H commute with T_F, then Proposition 2 implies that there exist two constants c_1 and c_2 such that $T_F^q = c_1 T_G^p$ and $T_F^s = c_2 T_H^p$. Now let m and n be two positive integers such that $mq = ns$. Hence $T_F^{mq} = T_F^{ns}$ and so

$$\left(c_1^{\frac{1}{p}} T_G\right)^{pm} = \left(c_2^{\frac{1}{p}} T_H\right)^{pn}.$$

Thus Proposition 3 implies that T_G commutes with T_H. □

Corollary 1. *If T_f and T_g are two Toeplitz operators with bounded symbols which commute with a quasihomogeneous Toeplitz operator, then they commute with each other.*

Proof. Let the polar decompositions of f and g be $\sum_{-\infty}^{\infty} e^{ik\theta} f_k(r)$ and $\sum_{-\infty}^{\infty} e^{il\theta} g_l(r)$ respectively. By Theorem LZ, each of the Toeplitz operators with symbols $e^{ik\theta} f_k$ and $e^{il\theta} g_l$ will commute with the quasihomogeneous Toeplitz operator for all $k, l \geq 0$. The negative parts in the polar decomposition of f and g are equal to zero by the same theorem. Now our main result, Theorem 1, implies that $T_{e^{ik\theta} f_k}$ and $T_{e^{il\theta} g_l}$ commute with each other for every k and l, hence T_f and T_g commute. □

Concluding remarks. In all the instances that we are aware of, if two Toeplitz operators commute with a third one, of course none of them being the identity, then they commute with each other. For example, non-trivial Toeplitz operators with radial symbols commute only with other such operators and non-trivial analytic Toeplitz operators commute only with other such operators. We are firmly convinced that this is a general fact about Toeplitz operators in the Bergman space and we conjecture the following: *If two Toeplitz operators commute with a third one, none of them being the identity, then they commute with each other.*

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I. Louhichi and N. V. Rao, The University of Toledo, College of Arts and Sciences, Department of Mathematics, Mail Stop 942, Toledo, Ohio 43606-3390, USA
e-mail: `issam.louhichi@gmail.com`
`rnagise@math.utoledo.edu`

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