Two questions on products of Toeplitz operators on the Bergman space

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Abstract. The zero product problem and the commuting problem for Toeplitz operators on the Bergman space over the unit disk are some of the most interesting unsolved problems. For bounded harmonic symbols these are solved but for general bounded symbols it is still far from being complete. This paper shows that the zero product problem holds for a special case where one of the symbols has certain polar decomposition and the other is a general bounded symbol. We also prove that the commutant of $T_{z+\overline{z}}$ is sum of powers of itself.

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1. Introduction

Let $dA = \frac{1}{\pi} r dr d\theta$, where (r, θ) are the polar coordinates in the complex plane $\mathbb{C}$, denote the normalized Lebesgue area measure on the unit disk $\mathbb{D}$, so that the measure of $\mathbb{D}$ equals 1. The Bergman space $L^2_a(\mathbb{D})$ is the Hilbert space consisting of analytic functions which are contained in $L^2(\mathbb{D}, dA)$. We denote the inner product in $L^2(\mathbb{D}, dA)$ by $\langle , \rangle$. It is well known that $L^2_a(\mathbb{D})$ is a closed subspace of the Hilbert space $L^2(\mathbb{D}, dA)$, and has the set $\{\sqrt{n+1} z^n \mid n \geq 0\}$ as an orthonormal basis. Let P be the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto $L^2_a(\mathbb{D})$. For a function $f \in L^\infty(\mathbb{D}, dA)$, the Toeplitz operator T_f with symbol f is the operator on $L^2_a(\mathbb{D})$ defined by $T_f h = P(fh)$, for $h \in L^2_a(\mathbb{D})$.

Two questions arise in this paper:

(Q1) For which bounded symbols f and g, would $T_f T_g = 0$ imply $T_f = 0$ or T_g=0, i.e., $f = 0$ or $g = 0$? In the case where this is true, we say that the zero product has only trivial solution.

(Q2) Assuming only one of the two symbols is harmonic that is neither analytic nor conjugate analytic, under which conditions $T_f T_g = T_g T_f$? This question is one of the open problems suggested in [3].

The corresponding questions for analogous Toeplitz operators defined on the Hardy space of the unit circle $\mathbb{T}$ was resolved by Brown and Halmos in their seminal paper [4]. They prove that the zero product has only trivial solution, and that $T_f T_g = T_g T_f$ for some f and g in $L^\infty(\mathbb{T})$ if and only if

(a) f and g are both analytic,
or
(b) $\bar{f}$ and $\bar{g}$ are both analytic,
or
(c) one of the two symbols is a linear function of the other.

On the Bergman space the above two questions are far from resolved. Concerning the question (Q1), Ahern and Čučković [1] show that the zero product has only trivial solution when both symbols are bounded harmonic functions. Later in [5], Čučković proves that if $f \in L^\infty(\mathbb{D}, dA)$ such that $T_f T_{z^j - \bar{z}^l} = 0$ where j and l are both positive integers, then $T_f = 0$. Moreover, he conjectures that if $f \in L^\infty(\mathbb{D}, dA)$ and g is a bounded harmonic on $\mathbb{D}$, then $T_f T_g = 0$ has only trivial solution. In section 3 of this paper, we present our contribution to solving this conjecture.

Regarding the question (Q2), in the current literature we did not find any work that treats the problem. In section 4 of this paper, we give a partial answer to this question.

2. Preliminaries

A function f is said to be quasihomogeneous of degree p an integer if it is of the form $e^{ip\theta}\phi$, where ϕ is a radial function. In this case the associated Toeplitz operator T_f is also called quasihomogeneous Toeplitz operator of degree p. Those Toeplitz operators were studied in [7], [11], [12], [9], [10] and [6]. The reason that we study such family of symbols is that any function f in $L^2(\mathbb{D}, dA)$ has the following polar decomposition

$$f(re^{i\theta}) = \sum_{k\in\mathbb{Z}} e^{ik\theta} f_k(r),$$

where f_k are radial functions in $L^2([0,1], rdr)$.

Before we state our results, we need to introduce the Mellin transform which is going to be our main tool. The Mellin transform $\widehat{f}$ of a radial function f in $L^1([0,1], rdr)$ is defined by

$$\widehat{f}(z) = \int_0^1 f(r) r^{z-1}\, dr.$$

It is well known that, for these functions, the Mellin transform is well defined on the right half-plane $\{z : \Re z \geq 2\}$ and it is analytic on $\{z : \Re z > 2\}$. It is important and helpful to know that the Mellin transform $\widehat{f}$ is uniquely determined by its values on any arithmetic sequence of integers. In fact we have the following classical theorem [14, p.102].

Theorem 1. *Suppose that f is a bounded analytic function on $\{z : \Re z > 0\}$ which vanishes at the pairwise distinct points $z_1, z_2 \cdots$, where*

i) $\inf\{|z_n|\} > 0$
and
ii) $\sum_{n\geq 1} \Re(\frac{1}{z_n}) = \infty$.

Then f vanishes identically on $\{z : \Re z > 0\}$.

Remark 1. Now one can apply this theorem to prove that if $f \in L^1([0,1], rdr)$ and if there exist $n_0, p \in \mathbb{N}$ such that

$$\widehat{f}(pk + n_0) = 0 \text{ for all } k \in \mathbb{N},$$

then $\widehat{f}(z) = 0$ for all $z \in \{z : \Re z > 2\}$ and so $f = 0$.

A direct calculation gives the following lemma which we shall use often.

Lemma 1. *Let $k, p \in \mathbb{N}$ and let f be an integrable radial function. Then*

$$T_{e^{ip\theta}f}(z^k) = 2(k+p+1)\widehat{f}(2k+p+2)z^{k+p}$$

and

$$T_{e^{-ip\theta}f}(z^k) = \begin{cases} 0 & \text{if } 0 \leq k \leq p-1 \\ 2(k-p+1)\widehat{f}(2k-p+2)z^{k-p} & \text{if } k \geq p. \end{cases}$$

Proof. For $p \geq 0$ and all $k \geq 0$, we have

$$\begin{aligned} T_{e^{ip\theta}f}(z^k) &= P(e^{ip\theta}fz^k) = \sum_{n\geq 0}(n+1)\langle e^{ip\theta}fz^k, z^n\rangle z^n \\ &= \sum_{n\geq 0}(n+1)\int_0^1\int_0^{2\pi} f(r)r^{k+n+1}e^{i(k+p-n)\theta}\frac{d\theta}{\pi}drz^n \\ &= 2(k+p+1)\widehat{f}(2k+p+2)z^{k+p}. \end{aligned}$$

A similar calculation gives the values of the quasihomogeneous Toeplitz operator of negative degree on the elements of the basis of $L_a^2(\mathbb{D})$. □

3. The zero product

The next proposition is an improvement of [5, Theorem 2, p. 237] with a completely different proof. In fact we use a recent result about finite rank Toeplitz operators due to Luecking [13]. This result can be stated as follows: *The only finite rank Toeplitz operator is the zero operator.*

Recently we come to know that Proposition 1 has been obtained independently by Trieu Le [8] at about the same time as we did.

Proposition 1. *Let $f \in L^\infty(\mathbb{D}, dA)$ and let $g \in L^\infty(\mathbb{D}, dA)$ with polar decomposition $g(re^{i\theta}) = \sum_{k=-\infty}^{N} e^{ik\theta}g_k(r)$ where N is a positive integer. Assume $n_0 \geq 0$ to be the smallest integer such that $\widehat{g_N}(2n+N+2) \neq 0$ for all $n \geq n_0$. If $T_fT_g = 0$, then $f = 0$.*

Proof. A straightforward calculation, using Lemma 1, shows that for all $n \geq 0$

$$T_g(z^n) = 2(n+N+1)\widehat{g_N}(2n+N+2)z^{n+N} + \sum_{k=-n}^{N-1} 2(n+k+1)\widehat{g_k}(2n+k+2)z^{n+k}.$$

By hypothesis $\widehat{g_N}(2n_0+N+2) \neq 0$, then

$$z^{n_0+N} \in \operatorname{Span}\{T_g(z^{n_0}), 1, z, \ldots, z^{n_0+N-1}\}. \tag{1}$$

Redoing the same argument, one can see that

$$z^{n_0+N+1} \in \operatorname{Span}\{T_g(z^{n_0+1}), 1, z, \ldots, z^{n_0+N}\}.$$

Then, using (1),

$$z^{n_0+N+1} \in \operatorname{Span}\{T_g(z^{n_0+1}), T_g(z^{n_0}), 1, z, \ldots, z^{n_0+N-1}\}.$$

In fact, the same method proves that for all $l \geq 0$

$$z^{n_0+N+l} \in \operatorname{Span}\{T_g(z^{n_0+l}), \ldots, T_g(z^{n_0}), 1, z, \ldots, z^{n_0+N-1}\}.$$

Because $T_fT_g = 0$,

$$T_f(z^{n_0+N+l}) \in \operatorname{Span}\{T_f(1), T_f(z), \ldots, T_f(z^{n_0+N-1})\}, \text{ for all } l \geq 0.$$

Therefore the rank of T_f is at most equal to $n_0 + N$. Hence, using Luecking's result, T_f must be zero and so $f = 0$. □

Remark 2. Since Mellin transforms of any monomial in $(z, \bar{z})$ are never equal to zero, then the hypothesis "$\widehat{g_N}(2n+N+2) \neq 0$ for all $n \geq n_0$" of Proposition 1 becomes superfluous if we take g to be equal to $P(z,\bar{z}) + \bar{\phi}$, where P is any polynomial in $(z, \bar{z})$ and ϕ is any bounded analytic function. This result supports the conjecture in [5, p. 235] stating that $T_fT_g = 0$, where $f \in L^\infty(\mathbb{D}, dA)$ and g harmonic, has only trivial solution.

4. Commutant of $T_{z+\bar{z}}$

In [2], Axler and Čučković prove that if f and g are both bounded harmonic in $\mathbb{D}$, then $T_fT_g = T_gT_f$ if and only if both symbols are analytic or both symbols are conjugate analytic or $af + bg$ is constant for some constants a, b not both zero. In other words when both symbols are bounded harmonic then the commutativity occurs only in trivial cases. Later with Rao [3], they prove that non-trivial Toeplitz operators with analytic symbols commute only with other such operators.

The question (Q2) is still an open problem even for "nice" bounded harmonic symbols such as $z + \bar{z}$. In this section, we describe the Toeplitz operator T_f that commutes with $T_{z+\bar{z}}$ when the symbol f is of the form $f(re^{i\theta}) = \sum_{k=-\infty}^{N} e^{ik\theta} f_k(r)$, where N is an integer.

Proposition 2. *Let $f(re^{i\theta}) = \sum_{k=-\infty}^{N} e^{ik\theta} f_k(r)$, where N is a positive integer, be a function in $L^1(\mathbb{D}, dA)$ such that the Toeplitz operator T_f is bounded. If T_f commutes with $T_{z+\bar{z}}$, then N must be less than or equal to 3.*

Proof. If T_f commutes with $T_{z+\bar{z}}$, then for all $n \geq 1$

$$T_f T_{z+\bar{z}}(z^n) = T_{z+\bar{z}} T_f(z^n).$$

Thus for all $n \geq 1$,

$$\sum_{k=-\infty}^{N} T_{e^{ik\theta} f_k} T_{z+\bar{z}}(z^n) = \sum_{k=-\infty}^{N} T_{z+\bar{z}} T_{e^{ik\theta} f_k}(z^n).$$

In the equation above, the term in z of degree $n+N+1$ on the left hand side comes from the product $T_{e^{iN\theta} f_N} T_z(z^n)$ only. On the right hand side only $T_z T_{e^{iN\theta} f_N}(z^n)$ provides the monomial z^{n+N+1}. Thus by equality we should have

$$T_{e^{iN\theta} f_N} T_z = T_z T_{e^{iN\theta} f_N}.$$

Since the symbol z is analytic then, by [3, p. 1952], $e^{iN\theta} f_N$ must be analytic too. This is possible if and only if $f_N = c_N r^N$ where c_N is a constant, i.e, $e^{iN\theta} f_N = c_N z^N$. Redoing the same argument for the terms in z of degree $n+N-1$ on both sides, gives us

$$c_N T_{z^N} T_{\bar{z}}(z^n) + T_{e^{i(N-2)\theta} f_{N-2}} T_z(z^n) = c_N T_{\bar{z}} T_{z^N}(z^n) + T_z T_{e^{i(N-2)\theta} f_{N-2}}(z^n), \text{ for all } n \geq 1,$$

which, by Lemma 1, is equivalent to

$$2(n+N)\widehat{f_{N-2}}(2n+N+2) - 2(n+N-1)\widehat{f_{N-2}}(2n+N) = c_N\left(\frac{n+N}{n+N+1} - \frac{n}{n+1}\right).$$

Now, using Remark 1, we obtain

$$2(z+N)\widehat{f_{N-2}}(2z+N+2) - 2(z+N-1)\widehat{f_{N-2}}(2z+N) = c_N\left(\frac{z+N}{z+N+1} - \frac{z}{z+1}\right), \text{ for } \Re z \geq 1. \quad (2)$$

Let F and G be the two bounded analytic functions in $\{z : \Re z > 1\}$ defined by $F(z) = 2(z+N-1)\widehat{f_{N-2}}(2z+N)$ and $G(z) = c_N\left(\frac{z+N-1}{z+N} + \frac{z+N-2}{z+N-1} + \ldots + \frac{z}{z+1}\right)$. Then equation (2) implies

$$F(z+1) - F(z) = G(z+1) - G(z), \text{ for } \Re z \geq 1.$$

Thus the function $F - G$ is periodic, and using [9, Lemma 6, p. 1428], we obtain $F(z) - G(z) = c$ where c is a constant. Therefore

$$\begin{aligned}
\widehat{f_{N-2}}(2z+N) &= \frac{c}{2z+2N-2} + \frac{c_N}{2z+2N-2} \sum_{j=1}^{N} \frac{z+N-j}{z+N+1-j} \\
&= \frac{1}{2z+2N} + \frac{c+c_N}{2z+2N-2} - \frac{2c_N}{(2z+2N-2)^2} \\
&\quad - c_N \sum_{j=3}^{N} \frac{1}{j-2}\left(\frac{1}{2z+2N+2-2j} - \frac{j-1}{2z+2N-2}\right).
\end{aligned}$$

Since $\widehat{r^m}(z) = \frac{1}{z+m}$ and $\widehat{r^m \ln r}(z) = -\frac{1}{(z+m)^2}$ for any integer m, then the above equality becomes

$$\begin{aligned}\widehat{f_{N-2}}(2z+N) &= c_N \widehat{r^N}(2z+N) + (c+c_N)\widehat{r^{N-2}}(2z+N) + 2c_N \widehat{r^{N-2}\ln r}(2z+N) \\ &\quad - c_N \sum_{j=3}^{N} \frac{1}{j-2}\big(\widehat{r^{N+2-2j}}(2z+N) - (j-1)\widehat{r^{N-2}}(2z+N)\big).\end{aligned}$$

Again Remark 1 implies

$$f_{N-2}(r) = c_N r^N + (c+N-1)r^{N-2} + 2c_N r^{N-2}\ln r - c_N \sum_{j=3}^{N} \frac{1}{j-2}\big(r^{N+2-2j} - (j-1)r^{N-2}\big).$$

If we assume $N \geq 4$, then the radial functions r^{N+2-2j} are in $L^1([0,1], rdr)$ if and only if $N - 2j > -4$ for all $3 \leq j \leq N$. In particular if $j = N$ then $N < 4$, which contradicts our assumption. Therefore N has to be less than or equal to 3. □

The following lemma is simple but crucial in the proof of the main result of this section.

Lemma 2. *The products $T^2_{z+\bar{z}}$ and $T^3_{z+\bar{z}}$ are both Toeplitz operators. Moreover*

$$T^2_{z+\bar{z}} = T_{z^2} + T_{1+\ln|z|^2} + T_{|z|^2} + T_{\bar{z}^2},$$

and

$$T^3_{z+\bar{z}} = T_{z^3} + T_{z(1+\ln|z|^2)} + T_{z^2\bar{z}} + T_{2z-\frac{1}{\bar{z}}} + T_{2\bar{z}-\frac{1}{z}} + T_{\bar{z}^2 z} + T_{\bar{z}(1+\ln|z|^2)} + T_{\bar{z}^3}.$$

Proof. Straightforward calculations using [11, Corollary 6.5, p. 533] imply that

$$T_z T_{\bar{z}} = T_{1+\ln|z|^2},$$

and

$$T_{z^2} T_{\bar{z}} = T_{2z-\frac{1}{\bar{z}}}.$$

Moreover, it is well known that $T_u T_v = T_{uv}$ whenever $\bar{u}$ is analytic or v is analytic. □

Remark 3. Here are some comments about Lemma 2.

i) The symbols $1+\ln|z|^2$, $z(1+\ln|z|^2)$ and $2z - \frac{1}{\bar{z}}$ and their conjugates, that appear in $T^3_{z+\bar{z}}$ and $T^2_{z+\bar{z}}$, obviously are not bounded but they are the so called "nearly bounded functions" [1, p.204]. Thus the Toeplitz operators associated to those symbols are all bounded.
ii) It is easy to see, again using [11, Corollary 6.5, p. 533], that $T^n_{z+\bar{z}}$ is not Toeplitz operator whenever $n \geq 4$.

Now, we are ready to state the main result of this section.

Theorem 2. *Let $f(re^{i\theta}) = \sum_{k=-\infty}^{N} e^{ik\theta} f_k(r)$ be a function in $L^1(\mathbb{D}, dA)$ such that the Toeplitz operator T_f is bounded. If T_f commutes with $T_{z+\bar{z}}$, then $T_f = Q(T_{z+\bar{z}})$ where Q is a polynomial of degree at most 3.*

Proof. Since T_f commutes with $T_{z+\bar{z}}$, then Proposition 2 implies that $N \leq 3$. So that $f(re^{i\theta}) = \sum_{k=-\infty}^{3} e^{ik\theta} f_k(r)$. As in the proof of Proposition 2, we show that $T_{e^{3i\theta} f_3} = c_3 T_{z^3}$ where c_3 is a constant. By Lemma 2, $T^3_{z+\bar{z}}$ is sum of quasihomogeneous Toeplitz operators each of degree not equal to 2. Thus when we subtract $c_3 T^3_{z+\bar{z}}$ from T_f, $T_{e^{2i\theta} f_2}$ will be the quasihomogeneous Toeplitz operator of highest degree in the semi-finite sum $\sum_{k=-\infty}^{3} T_{e^{ik\theta} f_k} - c_3 T^3_{z+\bar{z}}$. Now, the operator $\sum_{k=-\infty}^{3} T_{e^{ik\theta} f_k} - c_3 T^3_{z+\bar{z}}$ also commutes with $T_{z+\bar{z}}$, so that for all $n \geq 0$

$$\Big(\sum_{k=-\infty}^{3} T_{e^{ik\theta} f_k} - c_3 T^3_{z+\bar{z}}\Big) T_{z+\bar{z}}(z^n) = T_{z+\bar{z}} \Big(\sum_{k=-\infty}^{3} T_{e^{ik\theta} f_k} - c_3 T^3_{z+\bar{z}}\Big)(z^n). \qquad (3)$$

In equation (3), terms in z of degree $n+3$ come, on the left hand side from $T_{e^{2i\theta} f_2} T_z(z^n)$, and on the right hand side from $T_z T_{e^{2i\theta} f_2}(z^n)$. Thus by equality we have

$$T_{e^{2i\theta} f_2} T_z(z^n) = T_z T_{e^{2i\theta} f_2}(z^n) \text{ for all } n \geq 0.$$

Again using [3, p. 1952], we conclude that $T_{e^{2i\theta} f_2} = c_2 T_z^2$ where c_2 is a constant. Now, $T^2_{z+\bar{z}}$ is sum of quasihomogeneous Toeplitz operators all of them are of degree different from 1. So that when we subtract $c_2 T^2_{z+\bar{z}}$ from $\sum_{k=-\infty}^{3} T_{e^{ik\theta} f_k} - c_3 T^3_{z+\bar{z}}$, the operator $T_{e^{i\theta}(f_1 - c_3 \phi)}$, where $\phi(r) = r(1 + \ln r^2) + r^3 + (2r - \frac{1}{r})$, will be the only quasihomogeneous Toeplitz operator of degree 1 in $\sum_{k=-\infty}^{3} T_{e^{ik\theta} f_k} - c_3 T^3_{z+\bar{z}} - c_2 T^2_{z+\bar{z}}$. Here ϕ is the radial function in the symbol of the quasihomogeneous Toeplitz operator of degree 1 that appears in $T^3_{z+\bar{z}}$, which precisely is $T_{z(1+\ln|z|^2)} + T_{z^2\bar{z}} + T_{2z - \frac{1}{\bar{z}}}$, when it is written under the form $T_{e^{i\theta}\phi}$. Since $T_f - c_3 T^3_{z+\bar{z}} - c_2 T^2_{z+\bar{z}}$ commutes with $T_{z+\bar{z}}$, then by the same argument as before $T_{e^{i\theta}(f_1 - c_3\phi)}$ must commute with T_z, and so $T_{e^{i\theta}(f_1 - c_3\phi)} = c_1 T_z$ where c_1 is a constant. Hence $f_1 = c_1 r + c_3 \phi$. Redoing the same technique we see that in $\sum_{k=-\infty}^{3} T_{e^{ik\theta} f_k} - c_3 T^3_{z+\bar{z}} - c_2 T^2_{z+\bar{z}} - c_1 T_{z+\bar{z}}$, the quasihomogeneous Toeplitz operator of highest degree is $T_{(f_0 - c_2\psi)}$ where $\psi(r) = (1 + \ln r^2) + r^2$. In fact ψ is the symbol of the sum $T_{1+\ln|z|^2} + T_{|z|^2}$ that appears in $T^2_{z+\bar{z}}$. Also $T_f - c_3 T^3_{z+\bar{z}} - c_2 T^2_{z+\bar{z}} - c_1 T_{z+\bar{z}}$ commutes with $T_{z+\bar{z}}$, and so the same argument implies that $T_{f_0 - c_2\psi}$ commutes with T_z. Thus $T_{f_0 - c_2\psi} = c_0 I$, where I is the identity operator on $L^2_a(\mathbb{D})$. Hence $f_0 = c_0 + c_2\psi$. In $\sum_{k=-\infty}^{3} T_{e^{ik\theta} f_k} - c_3 T^3_{z+\bar{z}} - c_2 T^2_{z+\bar{z}} - c_1 T_{z+\bar{z}} - c_0 I$, the quasihomogeneous Toeplitz operator of highest degree is $T_{e^{-i\theta}(f_{-1} - c_1 r - c_3\phi)}$. Because $T_f - c_3 T^3_{z+\bar{z}} - c_2 T^2_{z+\bar{z}} - c_1 T_{z+\bar{z}} - c_0 I$ commutes with $T_{z+\bar{z}}$, then $T_{e^{-i\theta}(f_{-1} - c_1 r - c_3\phi)}$ must commute with T_z. However, since $T_{e^{-i\theta}(f_{-1} - c_1 r - c_3\phi)}$ is of quasihomogeneous degree -1, it cannot be analytic Toeplitz operator unless its symbol is zero i.e. $f_{-1} = c_1 r + c_3\phi$. In other words $\sum_{k=-\infty}^{3} T_{e^{ik\theta} f_k} - c_3 T^3_{z+\bar{z}} - c_2 T^2_{z+\bar{z}} - c_1 T_{z+\bar{z}} - c_0 I$ does not contain a quasihomogeneous Toeplitz operator of degree -1. In fact, by the same argument, we prove that in $\sum_{k=-\infty}^{3} T_{e^{ik\theta} f_k} - c_3 T^3_{z+\bar{z}} - c_2 T^2_{z+\bar{z}} - c_1 T_{z+\bar{z}} - c_0 I$ there are no Toeplitz operators of negative degree, and as a consequence we conclude that $f_{-2} = c_2 r^2$, $f_{-3} = c_3 r^3$ and $f_k = 0$ for all $k \leq -4$. Therefore, when we reconstitute the Toeplitz T_f, it appears that

$$\begin{aligned} T_f &= \sum_{k=-3}^{3} T_{e^{ik\theta} f_k} \\ &= c_3 T_{z^3} + c_2 T_{z^2} + c_1 T_z + c_3 T_{e^{i\theta}\phi} + c_0 I + c_2 T_\psi \\ &\quad + c_1 T_{\bar{z}} + c_3 T_{e^{-i\theta}\phi} + c_2 T_{\bar{z}^2} + c_3 T_{\bar{z}^3} \\ &= c_0 I + c_1 T_{z+\bar{z}} + c_2 T^2_{z+\bar{z}} + c_3 T^3_{z+\bar{z}}, \end{aligned}$$

where the last equality is obtained using Lemma 2. □

Remark 4. Using similar arguments, one can show that if f is as in Theorem 2 such that T_f commutes with $T_{z^2+\bar{z}}$, then $T_f = Q(T_{z^2+\bar{z}})$, where in this case Q is a polynomial of degree at most 2. Here, also using [11, Corollary 6.5, p. 533], it is easy to check that $T^k_{z^2+\bar{z}}$ is not a Toeplitz operator whenever $k \geq 3$, however $T^2_{z^2+\bar{z}}$ is a Toeplitz operator and $T^2_{z^2+\bar{z}} = T_{z^4} + T_{2z-\frac{1}{\bar{z}}} + T_{\bar{z}z^2} + T_{\bar{z}^2}$.

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