Research Statement

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July 18, 2007

My research focuses primarily on operator theory, functional analysis and complex analysis. In particular, I am interested in Toeplitz operators on the Bergman space $A^2(\mathbb{D})$ of the unit disc $\mathbb{D}$.

For a bounded function ϕ on $\mathbb{D}$, we denote by T_ϕ the Toeplitz operator on $A^2(\mathbb{D})$ with symbol ϕ, which is defined by :

$$\begin{array}{rcl} T_\phi : A^2(\mathbb{D}) & \longrightarrow & A^2(\mathbb{D}) \\ f & \longmapsto & P(\phi f)\ , \end{array}$$

where P denotes the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto $A^2(\mathbb{D})$.

Two natural questions arise in my research activity:

1) Under which conditions is the product of two Toeplitz operators a Toeplitz operator?

2) Under which conditions is the product of two Toeplitz operators commutative?

It is clear that if the symbol ϕ is a bounded analytic function on $\mathbb{D}$, then the Toeplitz operator T_ϕ is not other than the operator of multiplication by ϕ on $A^2(\mathbb{D})$ and these two questions become obvious. This we call the trivial case.

In a classical paper of Brown and Halmos [9], they give the answer to these two questions for the Toeplitz operator defined on the Hardy space $H^2(\mathbb{T})$ of the unit circle $\mathbb{T}$.
They show that for two bounded functions ϕ and ψ on $\mathbb{T}$:

i) *The product $T_\phi T_\psi$ is a Toeplitz operator T_ω if and only if either $\overline{\phi}$ is analytic or ψ is analytic. They also show that, in both cases $\omega = \phi\psi$.*
A function is called analytic if all its Fourrier coefficients of negative index are vanish.

ii) *The operators T_ϕ and T_ψ commute if and only if either ϕ and ψ are analytic, or $\overline{\phi}$ and $\overline{\psi}$ are analytic, or one is a linear function of the other.*

In short, in the Hardy space $H^2(\mathbb{T})$, the product of two Toeplitz operators is a Toeplitz operator (resp. two Toeplitz operators commute) only in the trivial case.

In the Bergman space, as usual, things are much more complicated, the questions above still resist to the specialists and are far from being solved.

1 On the product of Toeplitz operators on $A^2(\mathbb{D})$

In [2], Ahern and Čučković consider bounded harmonic symbols on $\mathbb{D}$ and prove that a theorem of Brown-Halmos type (this meaning that the symbol of the product is equal to the product of the symbols) is true only in the trivial case.

Let ϕ be a bounded harmonic function on $\mathbb{D}$, then there exists two functions ϕ_1 and ϕ_2 in the Bloch space such that

$$\phi = \phi_1 + \overline{\phi_2}, \tag{1}$$

and this decomposition is unique if $\phi_2(0) = 0$.
If we consider two bounded harmonic symbols $\phi = \phi_1 + \overline{\phi_2}$ and $\psi = \psi_1 + \overline{\psi_2}$ on $\mathbb{D}$ and we look to the product $T_\phi T_\psi$, it follows that :

$$\begin{aligned} T_\phi T_\psi &= T_{\phi_1} T_{\psi_1} + T_{\phi_1} T_{\overline{\psi_2}} + T_{\overline{\phi_2}} T_{\psi_1} + T_{\overline{\phi_2}} T_{\overline{\psi_2}} \\ &= T_{\phi_1 \psi_1} + T_{\phi_1} T_{\overline{\psi_2}} + T_{\overline{\phi_2} \psi_1} + T_{\overline{\phi_2 \psi_2}}. \end{aligned}$$

Thus if the product $T_{\phi_1} T_{\overline{\psi_2}}$ is a Toeplitz operator, then the product $T_\phi T_\psi$ will be also a Toeplitz operator. This is the essence of [1] in which Ahern characterizes, using the Berezin transform, all the holomorphic bounded functions ϕ and ψ such that the product $T_\phi T_{\overline{\psi}}$ is a Toeplitz operator. Consequently, Ahern gives a necessary and sufficient condition for a product of two Toeplitz operators with bounded harmonic symbols to be a Toeplitz operator.

Since the first question is solved for Toeplitz operators with harmonic symbols, one can ask what about the product of two Toeplitz operators with general symbols such bounded functions on $\mathbb{D}$? In a joint work with Elizabeth Strouse and Lova Zakariasy [13], we have studied the case of so-called quasihomogeneous symbols. A function f is said to be quasihomogeneous of degree p if $f(re^{ip\theta}) = e^{ip\theta}\phi(r)$ where ϕ is a radial function. In this case the associated Toeplitz operator T_f is also called quasihomogeneous Toeplitz operator of degree p. The reason to study such family of symbols is that any function f in $L^2(\mathbb{D}, dA)$ has the following polar decomposition :

$$f(re^{ip\theta}) = \sum_{k=-\infty}^{+\infty} e^{ik\theta} f_k(r)$$

where the functions f_k are in $L^2([0,1], rdr)$. We hope that the fact of having results on Toeplitz operators with quasihomogeneous symbols will enable us to say more on Toeplitz operators with general symbols.

Our techniques in [13] are completely different from those used in [2] and [1] since in our case the decomposition (1) is not valid any more. Instead the Berezin transform we use the Mellin transform to study Toeplitz operators. We obtain a necessary and sufficient condition for the product of two Toeplitz operators with quasihomogeneous symbols to be a Toeplitz operator too. Moreover we describe (when it is possible) the symbol of the product since this last symbol satisfies a Mellin convolution equation (which remains sometimes difficult to solve) involving the two first symbols.

Opens questions remain regarding the characterization of those pairs of Toeplitz operators with general symbols whose product is a Toeplitz operator.

2 On the commutativity of Toeplitz operators on $A^2(\mathbb{D})$

In [4], Axler and Čučković show that if ϕ and ψ are two bounded harmonic symbols (in this case the decomposition (1) is valid) and if T_ϕ commutes with T_ψ, then necessarily both ϕ and ψ are analytic, or both $\overline{\phi}$ and $\overline{\psi}$ are analytic, or one is a linear function of the other. Also, with Rao in [5], they prove that if ϕ is a bounded analytic function and if there exists a bounded function ψ such that T_ϕ and T_ψ commute, then ψ must be analytic too. As in the case of the product, one sees that the commutativity of two Toeplitz operators occurs only in the trivial case. In [15] and [16], Vasilevski gives the description of many (geometrically defined) classes of commuting Toeplitz operators. In [7], Čučković and Rao give a complete characterization of bounded Toeplitz operators which commute with Toeplitz operators with symbols $e^{ip\theta}r^m$ where both p and m are positive integers.

In a joint work with Lova Zakariasy [14], we are leaning in the question of the commutativity of Toeplitz operators with quasihomogeneous symbols. The results that we obtained can be summarized as follows: *Let p be a positive integer, ϕ be a nonzero bounded radial function and $f(re^{ik\theta}) = \sum_{k=-\infty}^{+\infty} e^{ik\theta} f_k(r)$ in $L^\infty(\mathbb{D}, dA)$. Then*

i) *T_f commutes with $T_{e^{ip\theta}\phi}$ if and only if $T_{e^{ik\theta}f_k}$ commutes with $T_{e^{ip\theta}\phi}$ for all $k \in \mathbb{Z}$.*

ii) *If there exists $k \leq -1$ such that $T_{e^{ik\theta}f_k}$ commutes with $T_{e^{ip\theta}\phi}$, then necessarily $f_k = 0$. This means that if T_f commutes with $T_{e^{ip\theta}\phi}$, then the negative part of index in the polar decomposition of f is vanish.*

iii) *If there exists $k \geq 0$ such that $T_{e^{ik\theta}f_k}$ commutes with $T_{e^{ip\theta}\phi}$, then f_k is unique up to constant factor. In particular f_0 is a constant.*

In [11], I am interested in the positive part index in the polar decomposition of f and it turns out that there is a relation between commutativity and properties of powers (then the products) of Toeplitz operators. In fact if $k \geq 0$ and if $T_{e^{ik\theta}f_k}$ and $T_{e^{ip\theta}\phi}$ commute, then $\big(T_{e^{ik\theta}f_k}\big)^p = c\big(T_{e^{ip\theta}\phi}\big)^k$, where c is a constant. Moreover, if k is a multiple of p, say for example $k = np$, then we obtain that $T_{e^{inp\theta}f_{np}} = c\big(T_{e^{ip\theta}\phi}\big)^n$. To motivate this characterization, I introduce a new notion-namely the $T-p^{th}$ *root* of Toeplitz operator. We say that a Toeplitz operator $T_{e^{ip\theta}\phi}$ has a T-p^{th} root if there exists a nonzero radial function ψ such that $T_{e^{ip\theta}\phi} = \big(T_{e^{i\theta}\psi}\big)^p$. The Toeplitz operator $T_{e^{ip\theta}r^m}$, studied in [7], has always a T-p^{th} root. Now we can state the main theorem of [11] as follows: *Let $T_{e^{ip\theta}\phi}$ be a bounded Toeplitz operator with quasihomogeneous degree $p \geq 1$. Assume that $T_{e^{ip\theta}\phi}$ has a T-p^{th} root $T_{e^{i\theta}\psi}$. Suppose that $f(re^{ip\theta}) = \sum_{k=0}^{+\infty} e^{ik\theta} f_k(r)$ is a bounded function such that T_f commutes with $T_{e^{ip\theta}\phi}$, then for all $k \geq 0$:*

i) *If $\big(T_{e^{i\theta}\psi}\big)^k$ is a Toeplitz operator, then either $T_{e^{ik\theta}f_k} = c\big(T_{e^{i\theta}\psi}\big)^k$ where c is a constant, or $f_k = 0$.*

ii) *If $\big(T_{e^{i\theta}\psi}\big)^k$ is not a Toeplitz operator, then $f_k = 0$.*

This theorem enables us to give an effective construction of **nontrivial symbols** (non analytic symbols) f such that $\big(T_f\big)^n$ is always a Toeplitz operator for all n in $\mathbb{N}$.

Recently, in a joint work with Rao [12], we improve the results in [11] to show a rare phenomenon, in the context of Toeplitz operators on the Bergman space, which can be stated as follows: *Commutant of a quasihomogeneous Toeplitz operator is equal to its bicommutant*. We recall that the commutant of an operator T is the set of all those operators that commute with it and bicommutant is the set of all operators that commute with all operators in the commutant. This work suggests several directions for further study that we hope to explore. In fact we conjecture that if two Toeplitz operators with general symbols (bounded symbols) commute with a third one, then they commute each other.

I am also interested in finite rank commutators and semicommutators of Toeplitz operators. For two Toeplitz operators T_ϕ and T_ψ we define the semicommutator and the commutator respectively by

$$(T_\phi, T_\psi] = T_{\phi\psi} - T_\phi T_\psi$$

and

$$[T_\phi, T_\psi] = T_\phi T_\psi - T_\psi T_\phi.$$

On the Hardy space H^2 of the unit disc $\mathbb{D}$, the question for which symbols ϕ and ψ the semicommutator $(T_\phi, T_\psi]$ or the commutator $[T_\phi, T_\psi]$ is of finite rank, has been completely solved in [3] and [17]. On the Bergman space, the zero semicommutator or commutator of two Toeplitz operators with harmonic symbols has been completely characterized in [4] and [18]. Recently, those results were generalized by Guo, Sun and Zheng [10]. In fact they proved that if the semicommutator or the commutator of two Toeplitz operators with bounded harmonic symbols has finite rank, then it must be zero. Unlike the harmonic case where finite rank semicommutators and commutators are trivial, together with Čučković [6], we prove that semicommutators and commutators of two quasihomogeneous Toeplitz operators of opposites degrees can be nonzero finite rank operators. Moreover we support our results by giving an effective constuction of nontrivial finite rank semicommutators and commutators of quasihomogeneous Toeplitz operators. Finding examples of nontrivial Toeplitz operators satisfying certain algebraic properties has been notoriously difficult, but such constructions are now possible using the recent results and techniques in [7], [13] and [14]. At the present we continue investigating on this problem which I believe contains many avenues for future work.

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