King Fahd University of Petroleum and Minerals
Department of Mathematical Sciences
Math201.01&02, **Final Exam**, Semester 051
Wed. Jan. 25, 2006 (12/25/ 1426)
Allowed Time: 3 Hours

1. Determine whether the vectors **u**=<3, 1, -1>, **v**=<0, 1, 2> and **w**=<1, 1, 5> lie in the same plane. (4 points)
2. Find the equation of the sphere with center (0, 1, 2) that is tangent to the plane $x+2y-3z=1$. (4 points)
3. Find the point of intersection of the two lines: (5 points)
$$L_1: x=2-t, y=2t, z=1+4t$$
$$L_2: x=2+t, y=3+4t, z=4+2t$$
4. Find the equation of the plane that passes through (-1, 0, 3) and is parallel to the plane $z=-4x+2y+1$. (4 points)
5. Describe the solid in ~~the~~ $3-space$ that is described in spherical coordinates by the inequalities $0\le\phi\le\frac{\pi}{2}$. (4 points)
6. (4 points each)
   a. Find $\lim_{(x,y)\to(0,0)}\frac{6x^2y}{x^4+2y^2}$
   b. Let $z=e^{x^2y}, x=\sqrt{u+v}, y=u^2+2$. Find $\frac{\partial z}{\partial u}|_{u=1,v=1}$.
   c. Let $f(x,y)=x^3y+x^2y^2-3y^4$. Show that $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=4f$.
7. Let $f(x,y)=\frac{x}{x+y}$. (4 points each)
   a. Find the largest possible value of the directional derivative of f at the point (1, 1) and determine in which direction does it occur.
   b. Find the equation of the tangent plane and the parametric equations of the normal line to the surface $z=f(x,y)$ at the point (1, 1, ½).
8. Find the absolute minimum and absolute maximum of $f(x,y)=x^2+2y^2-x-y$ over the triangular region R with vertices (0, 0), (1, 0) and (1, 1). (8 points)
9. Maximize $f(x,y)=x^2-2x+y^2+2y-1$ subject to the constraint $x^2-4x+y^2=-2$. (8 points)
10. Find the volume of the solid bounded by the surfaces $z=x^2+y^2+4, z=0, y=x^2$ and $y=4$. (7 points)
11. Evaluate the following double integrals: (7+7 points)
    a. $\int_0^8\int_{\sqrt[3]{y}}^2\cos(x^4)dxdy$.
    b. $\int_{-1}^1\int_0^{\sqrt{1-x^2}}e^{1-x^2-y^2}dydx$.
12. Evaluate $\iint_R\sin\theta\ dA$, where R is the region that is outside the curve $r=1+\cos\theta$ and inside the curve $r=3\cos\theta$. (7 points)

13. Set up, but do not evaluate, a triple integral for finding the volume of the solid lying in the first octant that is enclosed by the graphs of $z = 10 - 2x^2 - 2y^2$ and $z = 6$. (7 points)
14. Use spherical coordinates to evaluate $\iiint_G x \, dV$, where G is the solid lying in the first octant that is enclosed between the two surfaces $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 9$. (8 points)

All the best,
Dr. Ibrahim Al-Rasasi

Solution of Final Exam-051

(1) We need to Calculate the Scalar triple product of $\vec{u}, \vec{v}$ & $\vec{w}$:

$$\vec{u}\cdot(\vec{v}\times\vec{w}) = \begin{vmatrix} 3 & 1 & -1 \\ 0 & 1 & 2 \\ 1 & 1 & 5 \end{vmatrix} = 3(5-2) - 1(0-2) - 1(0-1) = 12$$

Since $\vec{u}\cdot(\vec{v}\times\vec{w}) \neq 0$, then the three given vectors do not lie in the same plane.

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(2) • Center $= (0, 1, 2)$

• Since the sphere is tangent to the plane $x+2y-3z-1=0$, then radius = distance from the center $(0,1,2)$ to the plane $x+2y-3z-1=0$

$$= \frac{|0+2(1)-3(2)-1|}{\sqrt{1^2+2^2+(-3)^2}} = \frac{5}{\sqrt{14}}$$

– So the equation of the sphere is

$$x^2 + (y-1)^2 + (z-2)^2 = \frac{25}{14}$$

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(3) If the two lines intersect, then there are two numbers t_1 & t_2 such that

$$\begin{cases} 2-t_1 = 2+t_2 & \sim (1) \\ 2t_1 = 3+4t_2 & \sim (2) \\ 1+4t_1 = 4+2t_2 & \sim (3) \end{cases}$$

• Solve first the system consisting of equations (1) & (2):

$2\times(1) + (2) \Rightarrow 4 = 7 + 6t_2 \Rightarrow t_2 = -\frac{1}{2}$

Substitute in (2) to get $2t_1 = 3-2 \Rightarrow t_1 = \frac{1}{2}$

• Next, check if $t_1 = \frac{1}{2}$ & $t_2 = -\frac{1}{2}$ satisfy (3).

$1+4\cdot\frac{1}{2} = 4+2\cdot(-\frac{1}{2}) \Rightarrow 3 = 3$. Ok.

So the two lines intersect. To find the point of intersection, substitute the value of t_1 in L_1 or t_2 in L_2:

$t_1 = \frac{1}{2} \xRightarrow{L_1} x = 2-\frac{1}{2} = \frac{3}{2}$, $y = 2-\frac{1}{2} = 1$, $z = 1+4\cdot\frac{1}{2} = 3$

The point of intersection is $(\frac{3}{2}, 1, 3)$.

[4] • Point on the plane is $(-1, 0, 3)$

• Since the required plane is parallel to the plane $4x - 2y + z = 1$, then the normal $\vec{n} = \langle 4, -2, 1 \rangle$ of the given plane is also normal to the required plane.

Thus, the equation of the required plane is

$$4(x+1) - 2(y-0) + 1(z-3) = 0$$

$$\Longrightarrow \quad 4x - 2y + z = -1$$

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[5] The Solid consists of all points P in 3-space such that the angle between the vector $\vec{OP}$ & the positive z-axis is $\phi \in [0, \frac{\pi}{2}]$.
Thus, the Solid is the upper half of the 3-space, including the xy-plane.
In set notation, the Solid is $\{(x, y, z) \in$ 3-space : $z \geq 0$, x and y are arbitrary$\}$.

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[6] a) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{6x^2y}{x^4+2y^2}$ $\quad \left(= \frac{0}{0}\text{, undefined}\right)$

• limit along C_1 : $y = 0$ (the x-axis)

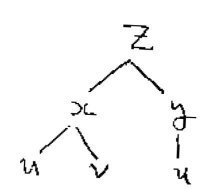
$$\lim_{\substack{(x,y)\to(0,0)\\ C_1}} \frac{6x^2y}{x^4+2y^2} = \lim_{x\to 0} \frac{0}{x^4+0} = \lim_{x\to 0} 0 = 0$$

• limit along C_2 : $y = x^2$ (parabola)

$$\lim_{\substack{(x,y)\to(0,0)\\ C_2}} \frac{6x^2y}{x^4+2y^2} = \lim_{x\to 0} \frac{6x^2 \cdot x^2}{x^4+2x^4} = \lim_{x\to 0} \frac{6x^4}{3x^4} = \lim_{x\to 0} 2 = 2$$

Since the limits along C_1 & C_2 are not equal, then the original limit does not exist.

b) $z = e^{x^2y}$, $x = \sqrt{u+v}$, $y = u^2 + 2$.



$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{dy}{du}$$

$$= 2xy\, e^{x^2y} \cdot \frac{1}{2\sqrt{u+v}} + x^2 e^{x^2y} \cdot 2u$$

$u=1,\ v=1 \Rightarrow x=\sqrt{1+1}=\sqrt{2}$ & $y=1^2+2=3$

$$\left.\frac{\partial z}{\partial u}\right|_{u=1,v=1} = 2\cdot\sqrt{2}\cdot 3\, e^{6}\cdot\frac{1}{2\sqrt{2}} + 2e^{6}\cdot 2\cdot 1 = 7e^{6}$$

c) $f(x,y)=x^3y+x^2y^2-3y^4$.

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y} = x\,(3x^2y+2xy^2)+y\,(x^3+2x^2y-12y^3)$$
$$= 3x^3y+2x^2y^2+x^3y+2x^2y^2-12y^4$$
$$= 4x^3y+4x^2y^2-12y^4$$
$$= 4\,(x^3y+x^2y^2-3y^4)$$
$$= 4\,f(x,y).$$

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7. $f(x,y)=\dfrac{x}{x+y}$.

$$f_x(x,y)=\frac{(x+y)\cdot 1 - x\cdot 1}{(x+y)^2} = \frac{y}{(x+y)^2}$$

$$f_y(x,y)=\frac{(x+y)\cdot 0 - x\cdot 1}{(x+y)^2} = \frac{-x}{(x+y)^2}$$

a) $\nabla f(x,y) = \langle f_x, f_y\rangle = \left\langle \dfrac{y}{(x+y)^2}, \dfrac{-x}{(x+y)^2}\right\rangle$

$\Rightarrow \nabla f(1,1) = \left\langle \frac{1}{4}, \frac{-1}{4}\right\rangle$.

The largest possible value of the directional derivative of f at $(1,1)$ is $\|\nabla f(1,1)\| = \sqrt{\frac{1}{16}+\frac{1}{16}} = \frac{\sqrt{2}}{4}$ & it occurs in the direction of the gradient vector $\nabla f(1,1)=\left\langle \frac{1}{4}, \frac{-1}{4}\right\rangle$.

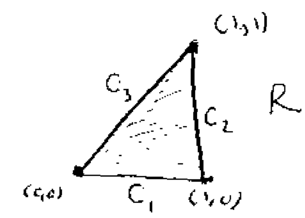
b) • Equation of the tangent plane at $(1,1,\frac{1}{2})$ is

$$Z = f(1,1)+f_x(1,1)(x-1)+f_y(1,1)(y-1)$$
$$\Rightarrow Z = \tfrac{1}{2}+\tfrac{1}{4}(x-1)-\tfrac{1}{4}(y-1)$$
$$\Rightarrow Z = \tfrac{1}{4}x-\tfrac{1}{4}y+\tfrac{1}{2}$$

• Parametric equations of the normal line at $(1,1,\frac{1}{2})$ are

$$\begin{aligned} x &= 1+f_x(1,1)\,t \\ y &= 1+f_y(1,1)\,t \\ z &= \tfrac{1}{2}-t \end{aligned} \quad\Rightarrow\quad \begin{aligned} x &= 1+\tfrac{1}{4}t \\ y &= 1-\tfrac{1}{4}t \\ z &= \tfrac{1}{2}-t \end{aligned}, \quad t\in(-\infty,+\infty).$$

[8] $f(x,y) = x^2 + 2y^2 - x - y$



- Critical points within R

  $f_x(x,y) = 2x - 1$

  $f_y(x,y) = 4y - 1$

  Both exist everywhere. So critical points come from the common zeros of f_x & f_y.

  $f_x = 0 \Rightarrow x = \frac{1}{2}$; $f_y = 0 \Rightarrow y = \frac{1}{4}$

  There is one critical point $(\frac{1}{2}, \frac{1}{4})$ within R.

- Boundary of R

  - C_1. $y = 0$, $0 \le x \le 1$

    $u(x) = f(x,0) = x^2 - x \Rightarrow u'(x) = 2x - 1 = 0 \Rightarrow x = \frac{1}{2}$

    $\Rightarrow$ points $(\frac{1}{2}, 0)$, $(0,0)$, $(1,0)$

  - C_2: $x = 1$, $0 \le y \le 1$

    $v(y) = f(1,y) = 2y^2 - y \Rightarrow v'(y) = 4y - 1 = 0 \Rightarrow y = \frac{1}{4}$

    $\Rightarrow$ points : $(1, \frac{1}{4})$, $(1,0)$, $(1,1)$

  - C_3 : $y = x$, $0 \le x \le 1$

    $w(x) = f(x,x) = 3x^2 - 2x \Rightarrow w'(x) = 6x - 2 = 0 \Rightarrow x = \frac{1}{3}$

    $\Rightarrow$ points. $(\frac{1}{3}, \frac{1}{3})$, $(0,0)$, $(1,1)$

- The values of f at the points found above

(x,y)	$(\frac{1}{2}, \frac{1}{4})$	$(\frac{1}{2}, 0)$	$(1, \frac{1}{4})$	$(\frac{1}{3}, \frac{1}{3})$	$(0,0)$	$(1,0)$	$(1,1)$
$f(x,y)$	$-\frac{3}{8}$ *	$-\frac{1}{4}$	$-\frac{1}{8}$	$-\frac{1}{3}$	0	0	1 *

The absolute minimum value of f is $-\frac{3}{8}$. It occurs at $(\frac{1}{2}, \frac{1}{4})$.

The absolute maximum value of f is 1. It occurs at $(1,1)$.

9 $f(x,y) = x^2 - 2x + y^2 + 2y - 1$, $g(x,y) = x^2 - 4x + y^2 + 2$

- $\nabla g(x,y) = \langle 2x-4, 2y \rangle = \vec{0} \Rightarrow x = 2$ & $y = 0$. Since $(2,0)$ does not lie on the constraint $x^2 - 4x + y^2 = -2$, then $\nabla g(x,y) \neq \vec{0}$ at all points on the constraint curve.

- We solve the system

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g = 0 \end{cases} \Rightarrow \begin{cases} 2x - 2 = \lambda \cdot (2x-4) & \sim (1) \\ 2y + 2 = \lambda \cdot 2y & \sim (2) \\ x^2 - 4x + y^2 + 2 = 0 & \sim (3) \end{cases}$$

(1) $\Rightarrow 2x - 2 = 2x\lambda - 4\lambda \Rightarrow 2x(1-\lambda) = 2(1-2\lambda) \Rightarrow x = \dfrac{1-2\lambda}{1-\lambda}$

($\lambda \neq 1$, otherwise (1) $\Rightarrow 2 = 4$, which is impossible)

(2) $\Rightarrow 2 = 2y(\lambda - 1) \Rightarrow y = \dfrac{1}{\lambda - 1} = \dfrac{-1}{1-\lambda}$

($\lambda \neq 1$, otherwise (2) $\Rightarrow 2 = 0$, which is impossible)

Substitute the values of x & y in (3):

$$\left(\frac{1-2\lambda}{1-\lambda}\right)^2 - 4\left(\frac{1-2\lambda}{1-\lambda}\right) + \left(\frac{-1}{1-\lambda}\right)^2 + 2 = 0$$

$\Rightarrow (1-2\lambda)^2 - 4(1-2\lambda)(1-\lambda) + 1 + 2(1-\lambda)^2 = 0$

$\Rightarrow 1 - 4\lambda + 4\lambda^2 - 4 + 12\lambda - 8\lambda^2 + 1 + 2 - 4\lambda + 2\lambda^2 = 0$

$\Rightarrow -2\lambda^2 + 4\lambda = 0 \Rightarrow 2\lambda(-\lambda + 2) = 0 \Rightarrow \underline{\underline{\lambda = 0}}$ or $\underline{\underline{\lambda = 2}}$

If $\lambda = 0$, then $(x,y) = (1,-1)$

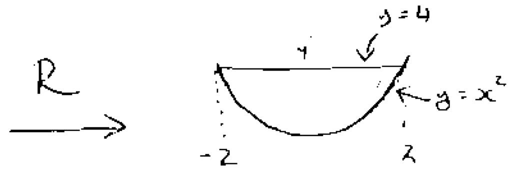
If $\lambda = 2$, then $(x,y) = (3,1)$

- $f(1,-1) = 1 - 2 + 1 - 2 - 1 = -3$

$f(3,1) = 9 - 6 + 1 + 2 - 1 = 5$

The maximum value of f is 5. It occurs at $(3,1)$.

$\boxed{10}$



Type I region

Using double Integrals,

$$V = \int_{-2}^{2}\int_{x^2}^{4} x^2+y^2+4 \;\; dy\, dx \;\; = \int_{-2}^{2} x^2y + \tfrac{1}{3}y^3 + 4y \Big]_{y=x^2}^{4} \; dx$$

$$= \int_{-2}^{2} (4x^2 + \tfrac{63}{4} + 16) - (x^4 + \tfrac{1}{3}x^6 + 4x^2) \;\; dx$$

$$= \int_{-2}^{2} \frac{127}{4} - x^4 - \tfrac{1}{3}x^6 \;\; dx \;\; \overset{\text{even}}{=} \; 2\int_{0}^{2} \frac{127}{4} - x^4 - \tfrac{1}{3}x^6 \; dx$$

$$= 2 \cdot \; \frac{127}{4}x - \tfrac{1}{5}x^5 - \tfrac{1}{27}x^7 \Big]_{0}^{2} \; = 2\left(\frac{127}{2} - \frac{32}{5} - \frac{128}{27}\right).$$

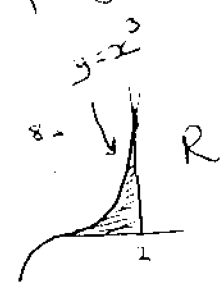
$\boxed{11}$ a) $\displaystyle\int_{0}^{8}\int_{\sqrt[3]{y}}^{2} \cos(x^4)\, dx\, dy.$

As it is not easy to integrate, we try to reverse the order of Integration.

Region: left Curve : $x = \sqrt[3]{y} \;\; \Rightarrow y = x^3$

right Curve , $x = 2$

y-Interval : $[0, 8]$.



Looking at the region as a type I region, we get

$$\int_{0}^{8}\int_{\sqrt[3]{y}}^{2} \cos(x^4)\, dx\, dy \;\; = \int_{0}^{2}\int_{0}^{x^3} \cos(x^4)\;\; dy\;\; dx$$

$$= \int_{0}^{2} \;\; y\cos(x^4)\Big]_{y=0}^{x^3} \; dx$$

$$= \int_{0}^{2} x^3\cos(x^4) \;\; dx$$

$$= \tfrac{1}{4}\sin(x^4)\Big]_{x=0}^{2}$$

$$= \frac{\sin(16)}{4}.$$

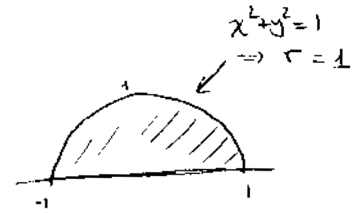
b) $\displaystyle\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} e^{1-x^2-y^2}\,dy\,dx$.

As it is not easy to integrate, we try polar coordinates.

Region: Lower Curve : $y=0$

Upper Curve : $y=\sqrt{1-x^2}$ (upper part of the unit circle $x^2+y^2=1$)

x-interval : $[-1,1]$



Using Polar Coordinate,

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} e^{1-x^2-y^2}\,dy\,dx = \int_0^{\pi}\int_0^1 e^{1-r^2}\cdot r\,dr\,d\theta$$

$$= \int_0^{\pi} -\tfrac{1}{2}\,e^{1-r^2}\Big]_{r=0}^{1}\,d\theta$$

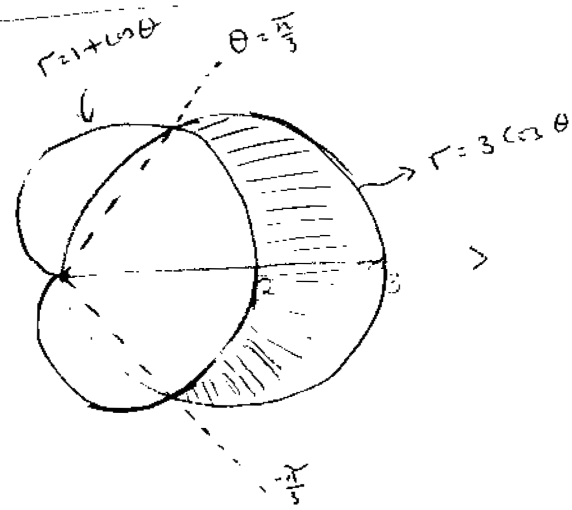
$$= \int_0^{\pi} -\tfrac{1}{2}(1-e)\,d\theta = \frac{\pi}{2}(e-1).$$

$\boxed{12}$ points of intersection.

$1+\cos\theta = 3\cos\theta$

$\Rightarrow \cos\theta = \frac{1}{2}$

$\Rightarrow \theta = -\frac{\pi}{3}, \frac{\pi}{3}$



$$\iint_R \sin\theta\,dA = \int_{-\pi/3}^{\pi/3}\int_{1+\cos\theta}^{3\cos\theta} \sin\theta\cdot r\,dr\,d\theta$$

$$= \int_{-\pi/3}^{\pi/3} \sin\theta\cdot\tfrac{1}{2}r^2\Big]_{r=1+\cos\theta}^{3\cos\theta}\,d\theta$$

$$= \int_{-\pi/3}^{\pi/3} \tfrac{1}{2}\sin\theta\left[9\cos^2\theta-(1+\cos\theta)^2\right]d\theta$$

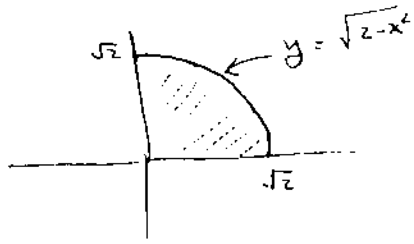
$= 0$, since the integrand is an odd function.

13

The projection in the xy-plane:

$$10 - 2x^2 - 2y^2 = 6$$
$$\Rightarrow 2x^2 + 2y^2 = 4$$
$$\Rightarrow x^2 + y^2 = 2 \quad , \text{ a Circle}$$

$R \Longrightarrow$



a type I region

$$V = \iiint_G 1 \, dV$$
$$= \int_0^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} \int_6^{10-2x^2-2y^2} 1 \; dz \, dy \, dx$$

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14 $x^2+y^2+z^2 = 1 \Rightarrow \rho = 1$ & $x^2+y^2+z^2 = 9 \Rightarrow \rho = 3$ (Both are spheres centered at the origin)

The solid is described in spherical coordinates by.

$$1 \le \rho \le 3, \quad 0 \le \phi \le \frac{\pi}{2}, \quad 0 \le \theta \le \frac{\pi}{2} .$$

$$\iiint_G x \, dV = \int_0^{\pi/2} \int_0^{\pi/2} \int_1^3 \rho \sin\phi \cos\theta \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$
$$= \int_0^{\pi/2} \int_0^{\pi/2} \int_1^3 \rho^3 \sin^2\phi \cos\theta \; d\rho \, d\phi \, d\theta$$
$$= \int_0^{\pi/2} \int_0^{\pi/2} \frac{1}{4}\rho^4 \Big]_{\rho=1}^{3} \sin^2\phi \cos\theta \; d\phi \, d\theta$$
$$= \int_0^{\pi/2} \int_0^{\pi/2} 20 \cdot \frac{1}{2}(1 - \cos(2\phi)) \cdot \cos\theta \; d\phi \, d\theta$$
$$= \int_0^{\pi/2} 10 \left(\phi - \frac{1}{2}\sin(2\phi)\right)\Big]_{\phi=0}^{\frac{\pi}{2}} \cdot \cos\theta \; d\theta$$
$$= \int_0^{\pi/2} 5\pi \cdot \cos\theta \; d\theta$$
$$= 5\pi \cdot \sin\theta \Big]_0^{\pi/2} = 5\pi .$$
