King Fahd University of Petroleum and Minerals
Department of Mathematical Sciences
Math201.01&02, **Exam II**, Semester 051
Monday, Dec. 12, 2005 (10/11/1426)
Allowed Time: 2 Hours

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1. Let $P = (2,-1,2)$ and $Q = (1,2,4)$ be two points in $3-space$. (6 points each)
   a. Find the parametric equations of the line L passing through P and Q.
   b. Where does the line L intersect the cylinder $x^2 + y^2 = 25$.
   c. Find the distance from the origin to the line L.

2. Find the equation of the plane that passes through $P = (1,0,-1)$ and $Q = (2,-1,1)$ and is perpendicular to the plane $z = 2x - y + 3$. (9 points)
3. (5 points each)
   a. Find the trace of the surface $z^3 - x^2 + y = 5$ in the plane $z = 2$.
   b. Identify and sketch the graph of the equation $z + 4x - 2x^2 - 3y^2 = 0$.
   c. Describe the graph of $\rho = 4\sin\theta\sin\phi$.
4. (8+6 points)
   a. Find the domain of $f(x,y) = \dfrac{y\sqrt{x^2 - 9}}{y+3}$. Write your answer in set notation. Sketch the domain.
   b. Let $f(x,y,z) = 3x - y + 2z + 1$. Find the level surface of f that passes through the point (3, 2, 1). Describe this level surface.
5. Find the following limits: (8 points each)
   a. $\lim_{(x,y)\to(1,1)} \dfrac{2-\sqrt{xy+3}}{xy-1}$.
   b. $\lim_{(x,y)\to(0,0)} \dfrac{\sin(2y^2)}{3x^2 + y^2}$.

6. Let $f(x,y) = \dfrac{y}{x^2 + y^2}$. (3+7 points)
   a. Find the slope of the surface $z = f(x,y)$ in the $x-direction$ at (1, 2).
   b. Find $\dfrac{\partial^3 f}{\partial x^2 \partial y}$. Simplify your answer.
7. Let $f(x,y) = 2x^2 y - x + 3y^2 + 1$. Use Differentials to approximate the change in the value of f as (x,y) moves from (1, 2) to (1.01, 1.99). (8 points)
8. Let $w = f(\frac{x}{y}, \frac{y}{z}, \frac{z}{x})$. Find the value of $x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} + z\dfrac{\partial w}{\partial z}$. (10 points)

All the Best,
Dr. Ibrahim Al-Rasasi.

Solution of Exam II

[1] $P = (2,-1,2)$, $Q = (1,2,4)$

a) Point $P = (2,-1,2)$, Vector parallel to the line L is $\overrightarrow{PQ} = \langle -1, 3, 2 \rangle$

The parametric equations of L are

$$x = 2 - t$$
$$y = -1 + 3t \qquad , \quad -\infty < t < +\infty$$
$$z = 2 + 2t$$

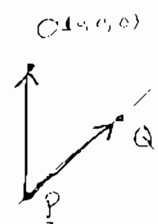
b) $(2-t)^2 + (-1+3t)^2 = 25 \Rightarrow t^2 - t - 2 = 0 \Rightarrow (t-2)(t+1) = 0$

$\Rightarrow t = -1, 2$.

When $t = -1$, then $(x,y,z) = (3,-4,0)$ } the points of intersection

$t = 2$, then $(x,y,z) = (0,5,6)$ } between the cylinder & L.

c) $d = \dfrac{\| \overrightarrow{PC} \times \overrightarrow{PQ} \|}{\| \overrightarrow{PQ} \|}$, $\overrightarrow{PC} = \langle -2, 1, -2 \rangle$



$$\overrightarrow{PC} \times \overrightarrow{PQ} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 1 & -2 \\ -1 & 3 & 2 \end{vmatrix} = 8\vec{i} + 6\vec{j} - 5\vec{k}$$

$$\Rightarrow d = \frac{\sqrt{64+36+25}}{\sqrt{1+9+4}} = \frac{\sqrt{125}}{\sqrt{14}} = \frac{5\sqrt{5}}{\sqrt{14}} = \frac{5}{14}\sqrt{70}$$

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[2] • $\overrightarrow{PQ} = \langle 1, -1, 2 \rangle$

• the normal to the given plane is $\vec{n} = \langle 2, -1, -1 \rangle$

• the normal to the required plane is

$$\vec{m} = \overrightarrow{PQ} \times \vec{n} = \begin{vmatrix} i & j & k \\ 1 & -1 & 2 \\ 2 & -1 & -1 \end{vmatrix} = 3i + 5j + k$$

• a point on the required plane is $P = (1,0,-1)$.

The equation of the required plane is

$$3(x-1) + 5(y-0) + 1(z+1) = 0$$

$$\Rightarrow 3x + 5y + z = 2$$

3 a) To find the trace, we solve the system

$$\begin{cases} z^3 - x^2 + y = 5 \\ z = 2 \end{cases}$$

$$\Rightarrow 2^3 - x^2 + y = 5 \Rightarrow y = x^2 - 3 \qquad (z = 2)$$

The trace is the parabola $y = x^2 - 3$ in the plane $z = 2$

b) $z + 4x - 2x^2 - 3y^2 = 0 \Rightarrow z = 2x^2 - 4x + 3y^2$

$$\Rightarrow z = 2(x-1)^2 + 3y^2 - 2$$

It is an elliptic paraboloid that results by translating the elliptic paraboloid $z = 2x^2 + 3y^2$ one unit along the positive x-axis and two units along the negative z-axis. That is, it is an elliptic paraboloid opening upward & with vertex $(1, 0, -2)$.

c) $\rho = 4 \sin\theta \sin\phi \Rightarrow \rho^2 = 4\rho \sin\theta \sin\phi$

$$\Rightarrow x^2 + y^2 + z^2 = 4y$$

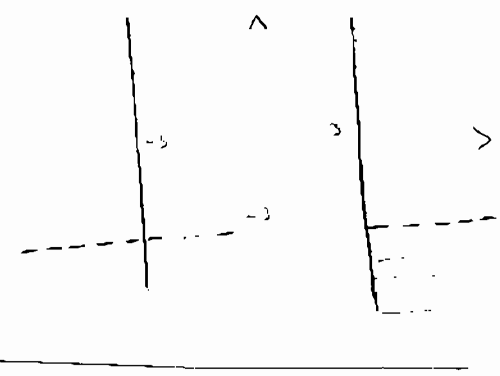
$$\Rightarrow x^2 + (y-2)^2 + z^2 = 4$$

The graph is a sphere with center $(0, 2, 0)$ and radius 2.

4 a) We must have $x^2 - 9 \geq 0$ & $y + 3 \neq 0$

$$\Rightarrow |x| \geq 3 \quad \& \quad y \neq -3$$

Domain $= \{(x, y) \text{ in 2-space} : |x| \geq 3 \ \& \ y \neq -3\}$.



b) The level surface of f is (height k)

$$3x - y + 2z + 1 = k$$

If the level surface passes through $(3, 2, 1)$, then $(3, 2, 1)$ satisfies the equation of the level surface:

$$3(3) - 2 + 2(1) + 1 = k$$

$$\Rightarrow k = 10$$

So the level surface is

$$3x - y + 2z + 1 = 10$$

or $3x - y + 2z = 9$

The level surface is a plane.

(3)

5 a) $\lim_{(x,y)\to(1,1)} \frac{2-\sqrt{xy+3}}{xy-1} = \lim_{(x,y)\to(1,1)} \frac{2-\sqrt{xy+3}}{xy-1} \cdot \frac{2+\sqrt{xy+3}}{2+\sqrt{xy+3}}$

$= \lim_{(x,y)\to(1,1)} \frac{4-(xy+3)}{(xy-1)\cdot 2+\sqrt{xy+3}} = \lim_{(x,y)\to(1,1)} \frac{-(xy-1)}{(xy-1)\cdot 2+\sqrt{xy+3}} = \frac{-1}{4}$

b) $\lim_{(x,y)\to(0,0)} \frac{\sin(2y^2)}{3x^2+y^2}$. $\frac{0}{0}$, undefined

along C_1 : $y=0$

$\lim_{\substack{(x,y)\to(0,0)\\C_1}} \frac{\sin(2y^2)}{3x^2+y^2} = \lim_{x\to0} \frac{\sin 0}{3x^2+0} = \lim_{x\to0} 0 = 0$

along C_2 : $y=x$

$\lim_{\substack{(x,y)\to(0,0)\\C_2}} \frac{\sin(2y^2)}{3x^2+y^2} = \lim_{x\to0} \frac{\sin(2x^2)}{4x^2} \overset{L.H}{=} \lim_{x\to0} \frac{\cos(2x^2)\cdot 4x}{8x} = \lim_{x\to0} \frac{1}{2}\cos(2x^2) = \frac{1}{2}$

Since the limits along C_1 & along C_2 are not equal, then the original limit does not exist.

6 $f(x,y) = \frac{4}{x^2+y^2}$

a). Slope $= f_x(1,2)$.

$f_x(x,y) = \frac{(x^2+y^2)\cdot 0 - 4(2x)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$

$\Longrightarrow$ Slope $= f_x(1,2) = \frac{-4}{25}$.

b) • $\frac{\partial f}{\partial y} = \frac{(x^2+y^2)\cdot 1 - y(2y)}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$

• $\frac{\partial^2 f}{\partial x \partial y} = \frac{(x^2+y^2)^2\cdot 2x - (x^2-y^2)\cdot 2(x^2+y^2)\cdot 2x}{(x^2+y^2)^4} = \frac{(x^2+y^2)\cdot 2x - 4x(x^2-y^2)}{(x^2+y^2)^3}$

$= \frac{6xy^2-2x^3}{(x^2+y^2)^3}$

• $\frac{\partial^3 f}{\partial x^2 \partial y} = \frac{(x^2+y^2)^3\cdot(6y^2-6x^2) - (6xy^2-2x^3)\, 3(x^2+y^2)^2\cdot 2x}{(x^2+y^2)^6}$

$= \frac{(x^2+y^2)(6y^2-6x^2) - 6x(6xy^2-2x^3)}{(x^2+y^2)^4}$

$= \frac{6(y^4-x^4) - 6(6x^2y^2-2x^4)}{(x^2+y^2)^4} = \frac{6(y^4-6x^2y^2+x^4)}{(x^2+y^2)^4}$

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[7] $f(x,y) = 2x^2y - x + 3y^2 + 1 \quad : \quad (1,2) \longrightarrow (1.01, 1.99)$

$$\Delta z \approx dz = f_x\,dx + f_y\,dy$$

$$= (4xy-1)\,dx + (2x^2+6y)\,dy$$

$dx = \Delta x = 1.01 - 1 = 0.01$

$dy = \Delta y = 1.99 - 2 = -0.01$

$$= (4(1)(2)-1)\cdot(0.01) + (2(1)^2+6(2))\cdot(-0.01)$$

$$= (0.01)(7 - 14) = -0.07$$

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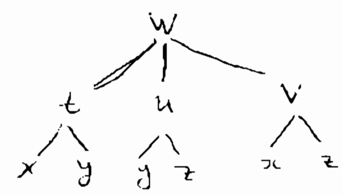
[8] $w = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$. Let $t = \frac{x}{y}$, $u = \frac{y}{z}$, $v = \frac{z}{x}$. Then

$w = f(t,u,v)$, $t = \frac{x}{y}$, $u = \frac{y}{z}$, $v = \frac{z}{x}$



- $\frac{\partial w}{\partial x} = \frac{\partial w}{\partial t}\cdot\frac{\partial t}{\partial x} + \frac{\partial w}{\partial v}\cdot\frac{\partial v}{\partial x} = \frac{\partial w}{\partial t}\cdot\frac{1}{y} + \frac{\partial w}{\partial v}\cdot\frac{-z}{x^2}$
- $\frac{\partial w}{\partial y} = \frac{\partial w}{\partial t}\cdot\frac{\partial t}{\partial y} + \frac{\partial w}{\partial u}\cdot\frac{\partial u}{\partial y} = \frac{\partial w}{\partial t}\cdot\frac{-x}{y^2} + \frac{\partial w}{\partial u}\cdot\frac{1}{z}$
- $\frac{\partial w}{\partial z} = \frac{\partial w}{\partial u}\cdot\frac{\partial u}{\partial z} + \frac{\partial w}{\partial v}\cdot\frac{\partial v}{\partial z} = \frac{\partial w}{\partial u}\cdot\frac{-y}{z^2} + \frac{\partial w}{\partial v}\cdot\frac{1}{x}$

- $x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z}$

$$= \left(\frac{\partial w}{\partial t}\cdot\frac{x}{y} + \frac{\partial w}{\partial v}\cdot\frac{-z}{x}\right) + \left(\frac{\partial w}{\partial t}\cdot\frac{-x}{y} + \frac{\partial w}{\partial u}\cdot\frac{y}{z}\right) + \left(\frac{\partial w}{\partial u}\cdot\frac{-y}{z} + \frac{\partial w}{\partial v}\cdot\frac{z}{x}\right)$$

= Zero.

Done