King Fahd University of Petroleum and Minerals
Department of Mathematical Sciences
Math201.01&02, **Exam I**, Semester 051
Tuesday, Oct. 11, 2005(8 Ramadan, 1426)
Allowed Time: 2 Hours

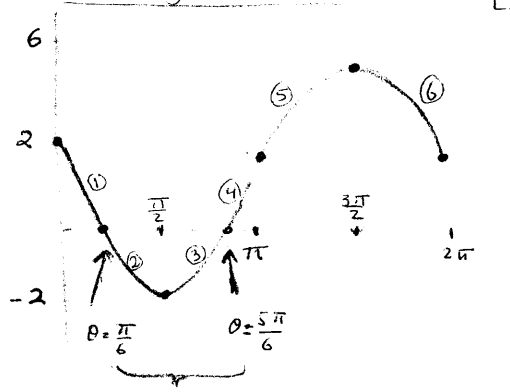
---

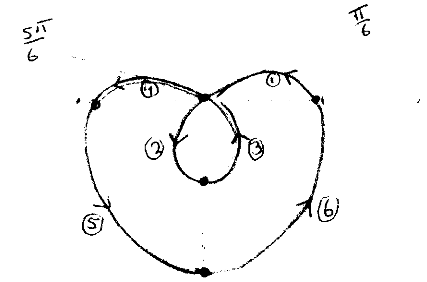
1. Consider the polar equation $r = 2 - 4\sin\theta$. (10+8 points)
   a. Sketch the graph of the given equation. Show the details.
   b. Find the area of the region enclosed by the inner loop of the graph.
2. Consider the polar curve whose equation is $r = 3\sin(5\theta)$. (8+8 points)
   a. Find the slope of the tangent line to the curve at the point where $\theta = \dfrac{\pi}{10}$.
   b. Find the equations of the tangent lines to the curve at the pole.
3. Find the area of the region inside the graph of $r = 2 + 2\cos\theta$ and outside the graph of $r = 3$. (10 points)
4. (6 points each)
   a. Convert the polar equation $\theta = \dfrac{2\pi}{3}$ to a Cartesian equation.
   b. Find the conditions on $a, b,$ and c that make the point (a, b, c) lies in the $yz-plane$.
5. A sphere has a diameter with endpoints (2, -2, 2) and (1, 1, -1). (8+4 points)
   a. Find the equation of this sphere.
   b. Find two points on this sphere different from the points (2, -2, 2) and (1, 1, -1).
6. Let **v**=**i**-**j**+2**k**, **w**=3**i**+5**j**-2**k**, and **z**=2**i**+4**k** be three vectors in 3-space. (5 points each except (f) where it is worth 7 points)
   a. Find the norm of 2**v**-**w**.
   b. Find a unit vector having the same direction as the vector **v**.
   c. Find a vector of magnitude 3 and having the opposite direction of **w**.
   d. Find a vector with initial point (-1, -1, -1) that is equivalent to **z**.
   e. Find the angle between the vectors **w** and **z**. Is the angle acute or obtuse.
   f. Find all values of a so that the vector $\mathbf{n} = \langle a, -3, a^2 \rangle$ is orthogonal to the vector **z**-**v**.

All the best,
Dr. Ibrahim Al-Rasasi.

1

a) Rectangular Plane $\boxed{r = 2 - 4\sin\theta}$ Polar Plane





$r = 0 \Rightarrow \sin\theta = \frac{1}{2}$

$\Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$

b) The Inner loop is graphed on the θ-interval : $\frac{\pi}{6} \le \theta \le \frac{5\pi}{6}$.

$$A = \int_{\frac{\pi}{6}}^{5\pi/6} \frac{1}{2}(2 - 4\sin\theta)^2\, d\theta$$

$$= 2\int_{\frac{\pi}{6}}^{\pi/2} \frac{1}{2}(2 - 4\sin\theta)^2\, d\theta \quad , \text{ by Symmetry}$$

$$= \int_{\frac{\pi}{6}}^{\pi/2} 12 - 16\sin\theta - 8\cos(2\theta)\; d\theta = 12\theta + 16\cos\theta - 4\sin(2\theta)\Big]_{\frac{\pi}{6}}^{\pi/2} = 4\pi - 6\sqrt{3}$$

2 $r = 3\sin(5\theta)$

a) $$\frac{dy}{dx} = \frac{r\cos\theta + \sin\theta\,\frac{dr}{d\theta}}{-r\sin\theta + \cos\theta\,\frac{dr}{d\theta}} = \frac{3\sin(5\theta)\cdot\cos\theta + \sin\theta\cdot 15\cos(5\theta)}{-3\sin(5\theta)\cdot\sin\theta + \cos\theta\cdot 15\cos(5\theta)}$$

$$\text{Slope} = \frac{dy}{dx}\Big|_{\theta = \frac{\pi}{10}} = \frac{3\cdot 1\cdot\cos(\frac{\pi}{10}) + \sin(\frac{\pi}{10})\cdot 15(0)}{-3\cdot 1\cdot\sin(\frac{\pi}{10}) + \cos(\frac{\pi}{10})\cdot 15(0)} = -\cot(\frac{\pi}{10})$$

b) The equations of the tangent lines to the curve at the pole are $\theta = \theta_0$ where θ_0 satisfies

$$r\Big|_{\theta=\theta_0} = 0 \quad \& \quad \frac{dr}{d\theta}\Big|_{\theta=\theta_0} \ne 0 \quad , \text{ Restrict } \theta_0 \text{ in the interval } [0, 2\pi].$$

. $r = 0 \Rightarrow 3\sin 5\theta = 0 \Rightarrow 5\theta = n\pi$ where n is integer

$\Rightarrow \theta = \frac{n}{5}\pi$

$\Rightarrow \theta = 0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}, \pi, \frac{6\pi}{5}, \frac{7\pi}{5}, \frac{8\pi}{5}, \frac{9\pi}{5}, 2\pi$

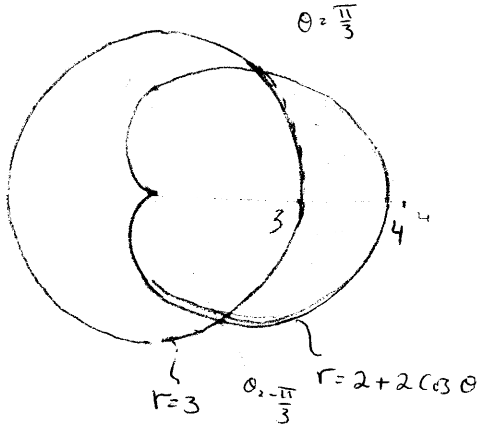
$\Rightarrow \theta = 0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}$ (the rest are repeated)

. It is very easy to check that $\frac{dr}{d\theta}\Big|_{\theta = 0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}} \ne 0$ $\quad \left[\frac{dr}{d\theta} = 15\cos(5\theta)\right]$

So the required tangent lines are

$\theta = 0$, $\theta = \frac{\pi}{5}$, $\theta = \frac{2\pi}{5}$, $\theta = \frac{3\pi}{5}$, $\theta = \frac{4\pi}{5}$.

3



points of intersection :

$3 = 2 + 2\cos\theta \implies \cos\theta = \frac{1}{2}$

$\implies \theta = \frac{\pi}{3}, -\frac{\pi}{3}$

By Symmetry, $A = 2 \cdot \frac{1}{2} \int_0^{\pi/3} (2+2\cos\theta)^2 - (3)^2 \, d\theta$

$= \int_0^{\pi/3} -3 + 8\cos\theta + 2\cos(2\theta) \, d\theta = -3\theta + 8\sin\theta + \sin(2\theta) \Big]_0^{\pi/3} = \frac{9\sqrt{3}}{2} - \pi$

4

a) $\theta = \frac{2\pi}{3}$. $\tan\theta = \frac{y}{x} \implies \tan(\frac{2\pi}{3}) = \frac{y}{x} \implies -\sqrt{3} = \frac{y}{x} \implies y = -\sqrt{3}\,x$.

b) The point (a,b,c) will lie in the yz-plane if and only if

$a = 0$

b is any real number

c is any real number.

5 End points of a diameter $(2,-2,2)$ & $(1,1,-1)$

a) Center = midpoint of the given diameter $= \left(\frac{2+1}{2}, \frac{-2+1}{2}, \frac{2-1}{2}\right) = \left(\frac{3}{2}, \frac{-1}{2}, \frac{1}{2}\right)$

radius $= \frac{1}{2} \cdot$ length of the diameter $= \frac{1}{2} \cdot \sqrt{(2-1)^2 + (-2-1)^2 + (2+1)^2} = \frac{1}{2}\sqrt{19}$

The equation of the sphere is

$$\left(x - \frac{3}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 + \left(z - \frac{1}{2}\right)^2 = \frac{19}{4}.$$

b) Two possible points on the sphere can be found by taking

$x = \frac{3}{2}$, $y = -\frac{1}{2}$ & $(z - \frac{1}{2})^2 = \frac{19}{4}$

$\implies x = \frac{3}{2}$, $y = -\frac{1}{2}$ & $z = \frac{1 \pm \sqrt{19}}{2}$

$\implies \left(\frac{3}{2}, -\frac{1}{2}, \frac{1+\sqrt{19}}{2}\right)$ & $\left(\frac{3}{2}, -\frac{1}{2}, \frac{1-\sqrt{19}}{2}\right)$.

6 $\vec{V} = \langle 1, -1, 2\rangle$, $\vec{W} = \langle 3, 5, -2\rangle$, $\vec{Z} = \langle 2, 0, 4\rangle$

a) $2\vec{V} - \vec{W} = \langle -1, -7, 6\rangle \Longrightarrow \|2\vec{V} - \vec{W}\| = \sqrt{(-1)^2 + (-7)^2 + 6^2} = \sqrt{86}$

b) The required vector is $\vec{u} = \dfrac{\vec{V}}{\|\vec{V}\|}$, $\|\vec{V}\| = \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6}$

$= \frac{1}{\sqrt{6}} \langle 1, -1, 2\rangle$

$= \langle \frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}} \rangle$

c) The required vector is $\vec{m} = -3 \dfrac{\vec{W}}{\|\vec{W}\|}$, $\|\vec{W}\| = \sqrt{3^2 + 5^2 + (-2)^2} = \sqrt{38}$

$= \frac{-3}{\sqrt{38}} \langle 3, 5, -2\rangle$

$= \langle \frac{-9}{\sqrt{38}}, \frac{-15}{\sqrt{38}}, \frac{6}{\sqrt{38}} \rangle$

d) a vector with initial point $P_0 = (-1, -1, -1)$ & terminal point $P_1 = (a, b, c)$ is equivalent to $\vec{Z}$;

$\overrightarrow{P_0P_1} = \vec{Z}$

$\Longrightarrow \langle a+1, b+1, c+1\rangle = \langle 2, 0, 4\rangle$

$\Longrightarrow \begin{cases} a+1 = 2 \\ b+1 = 0 \\ c+1 = 4 \end{cases} \Longrightarrow \begin{cases} a = 1 \\ b = -1 \\ c = 3 \end{cases}$

So the terminal point is $P_1 = (1, -1, 3)$

f) $\vec{Z} - \vec{V} = \langle 1, 1, 2\rangle$. If $\vec{n}$ is orthogonal to $\vec{Z} - \vec{V}$, then

$\vec{n} \cdot (\vec{Z} - \vec{V}) = 0$

$\Longrightarrow a - 3 + 2a^2 = 0$

$\Longrightarrow 2a^2 + a - 3 = 0$

$\Longrightarrow (2a+3)(a-1) = 0$

$\Longrightarrow a = 1$ or $a = -\frac{3}{2}$

e) $\vec{W} \cdot \vec{Z} = \|\vec{W}\| \|\vec{Z}\| \cos\theta \Longrightarrow -2 = \sqrt{38} \cdot \sqrt{20} \cos\theta \Longrightarrow \cos\theta = \dfrac{-2}{\sqrt{38} \cdot 2\sqrt{5}} = \dfrac{-1}{\sqrt{190}}$

$\Longrightarrow \theta = \cos^{-1}\left(\frac{-1}{\sqrt{190}}\right)$

Since $\cos\theta < 0$ & $0 \le \theta < \pi$ (by definition), then θ lies in the second quadrant & hence θ is an obtuse angle.