King Fahd University of Petroleum and Minerals
Department of Mathematical Sciences
Math201.03&04, **Exam II**, Semester 052
April 19, 2006, (19 Rabi' I, 1427)
Allowed Time: 100 minutes

1. Find the volume of the parallelpiped that has the vectors **u**=<1, 1, 3>, **v**=<2, 0, 2> and **w**=<1, -2, 1> as adjacent sides. (8 points)
2. Show that the following two lines are skew: (10 points)
$$L_1 : x = 1 + 5t, y = 3 + 6t, z = -7t$$
$$L_2 : x = 2 + t, y = -2 + t, z = 1 + 2t$$
3. Find the equation of the plane P that passes through (1, 2, 3) and is perpendicular to the planes $P_1 : x + 2y - z = 1$ and $P_2 : 3x - y + 4z = 5$. (10 points)
4. Find a point whose distance from the plane $2x - 3y + z = 4$ is 2 units. (8 points)
5. If a point Q has cylindrical coordinates $(2, \frac{\pi}{3}, a)$ and spherical coordinates $(b, c, \frac{\pi}{3})$, then find the rectangular coordinates of Q. (10 points)
6. (6+6 points)
   a. Describe the region in 3-space that satisfies the inequalities $1 \le r \le 4$, $0 \le z \le 5$.(Inequalities are given in cylindrical coordinates.)
   b. Find the equation of the surface that results when the surface $z = y^3 - x^3$ is reflected about the plane $x = 0$.
7. Let $f(x, y) = \sqrt{x^2 + y^2 - 4}$. (7+4+7)
   a. Find the domain of f (Give your answer in set notation.) and sketch the domain of f.
   b. Describe the level curve of height 3 of f.
   c. Identify and sketch the graph of f.
8. Find the following limits: (8+8 points)
   a. $\lim_{(x,y)\to(0,0)} \dfrac{x-2}{x^4 + y^6}$.
   b. $\lim_{(x,y)\to(0,0)} \dfrac{1-\cos(\sqrt{y})}{3x^2 - y}$.
9. Let $f(x, y) = \sin^{-1} x + \tan^{-1}(xy)$. Find $\dfrac{\partial^3 f}{\partial x^2 \partial y}$. Simplify your answer. (8 points)

All the best,
Dr. Ibrahim Al-Rasasi.

Solution of Exam II - 052

[1] Volume of the parallelpiped $= |\vec{u} \cdot \vec{v} \times \vec{w}|$

$$\vec{u} \cdot \vec{v} \times \vec{w} = \begin{vmatrix} 1 & 1 & 3 \\ 2 & 0 & 2 \\ 1 & -2 & 1 \end{vmatrix} = 1(0+4) - 1(2-2) + 3(-4-0) = -8$$

Volume $= |u \cdot v \times w| = |-8| = 8$ unit cubed.

[2] To show that L_1 & L_2 are skew, we need to show two things:

1. L_1 & L_2 are not parallel
2. L_1 & L_2 do not intersect.

1. L_1 & L_2 are not parallel. parallel vector to L_1 is $\vec{V}_1 = \langle 5, 6, -7 \rangle$

parallel vector to L_2 is $\vec{V}_2 = \langle 1, 1, 2 \rangle$

$$\vec{V}_1 \times \vec{V}_2 = \begin{vmatrix} i & j & k \\ 5 & 6 & -7 \\ 1 & 1 & 2 \end{vmatrix} = 19i - 17j - k$$

Since $\vec{V}_1 \times \vec{V}_2 \neq \vec{0}$, then $\vec{V}_1$ & $\vec{V}_2$ are not parallel and hence L_1 & L_2 are not parallel.

2. L_1 & L_2 do not intersect. We try to find the points of intersection.

$$1 + 5t_1 = 2 + t_2 \quad \text{---} (1)$$
$$3 + 6t_1 = -2 + t_2 \quad \text{---} (2)$$
$$-7t_1 = 1 + 2t_2 \quad \text{---} (3)$$

$(1) - (2) \Rightarrow -2 - t_1 = 4 \Rightarrow t_1 = -6$

Sub $t_1 = -6$ in (1) : $1 + 5(-6) = 2 + t_2 \Rightarrow t_2 = -31$

Check if $t_1 = -6$ & $t_2 = -31$ satisfy (3) :

$-7t_1 = 1 + 2t_2 \Rightarrow -7(-6) = 1 + 2(-31) \Rightarrow 42 = -61$ (NO)

Since $t_1 = -6$ & $t_2 = -31$ do not satisfy (3), then L_1 & L_2 do not intersect.

Conclusion. Since L_1 & L_2 are not parallel and do not intersect, then L_1 & L_2 are skew lines.

3. The required plane P: normal vector $= \vec{n}$
a point in P is $(1, 2, 3)$

plane P_1 : $x+2y-z=1 \Rightarrow$ normal vector is $\vec{n}_1 = \langle 1, 2, -1\rangle$
plane P_2 : $3x-y+4z=5 \Rightarrow$ normal vector is $\vec{n}_2 = \langle 3, -1, 4\rangle$

Now $P \perp P_1 \Rightarrow \vec{n} \perp \vec{n}_1$
$P \perp P_2 \Rightarrow \vec{n} \perp \vec{n}_2$.

So $\vec{n}$ is ~~per~~ orthogonal to both $\vec{n}_1$ and $\vec{n}_2$. Thus we can take
$\vec{n} = \vec{n}_1 \times \vec{n}_2$.

$$\vec{n} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ 3 & -1 & 4 \end{vmatrix} = 7i - 7j - 7k$$

The equation of the plane P is

$$7(x-1) - 7(y-2) - 7(z-3) = 0$$

$$\Rightarrow \boxed{x - y - z = -4}$$

4. We need to find a point (x_0, y_0, z_0) whose distance from the plane
$2x-3y+z=4$ is 2 units. That is,

$$2 = \frac{|2x_0 - 3y_0 + z_0 - 4|}{\sqrt{2^2 + (-3)^2 + 1^2}} = \frac{|2x_0 - 3y_0 + z_0 - 4|}{\sqrt{14}}$$

We can choose $x_0 = 0$, $y_0 = 0$, and $z_0 = 2\sqrt{14} + 4$

$$\Rightarrow (x_0, y_0, z_0) = (0, 0, 2\sqrt{14}+4)$$

There are other (infinitely many) choices.

5. Q has Cylindrical Coordinates $(2, \frac{\pi}{3}, a) = (r, \theta, z)$
spherical Coordinates $(b, c, \frac{\pi}{3}) = (\rho, \theta, \phi)$.

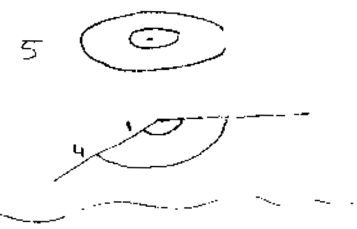
Required rectangular coordinates (x, y, z).

- $x = r\cos\theta = 2\cos\frac{\pi}{3} = 2 \cdot \frac{1}{2} = 1$
- $y = r\sin\theta = 2\sin\frac{\pi}{3} = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$
- $\tan\phi = \frac{r}{z} \Rightarrow \tan(\frac{\pi}{3}) = \frac{2}{z} \Rightarrow \sqrt{3} = \frac{2}{z} \Rightarrow z = \frac{2}{\sqrt{3}}$

Thus $(x, y, z) = (1, \sqrt{3}, \frac{2}{\sqrt{3}})$.

6 (a) $1 \le r \le 4$, $0 \le z \le 5$

It is the region in 3-space that lies between the two Cylinders $r=1$ and $r=4$ (the two Cylinders are included) & lying above the xy-plane ($z=0$) and below the plane $z=5$ (the two planes are included)



(b) $z = y^3 - x^3$

If this surface is reflected about the plane $x=0$, the equation of the new surface is given by ($x \longrightarrow -x$).

$$z = y^3 - (-x)^3$$

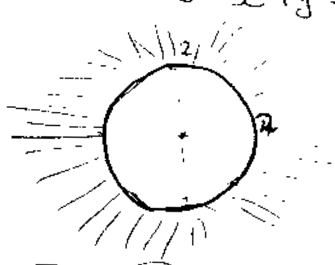
$$\Longrightarrow z = y^3 + x^3$$

7 $f(x,y) = \sqrt{x^2+y^2-4}$

a) We must have $x^2+y^2-4 \ge 0$, i.e., $x^2+y^2 \ge 4$.

Domain $= \{ (x,y) \text{ in 2-space} : x^2+y^2 \ge 4 \}$

In words, the domain consists of all points in the xy-plane lying on & outside the Circle $x^2+y^2=4$.



b) The level Curve of height 3 of f is the graph of the equation

$$f(x,y) = 3$$

$$\Longrightarrow \sqrt{x^2+y^2-4} = 3$$

$$\Longrightarrow x^2+y^2-4 = 9$$

$$\Longrightarrow x^2+y^2 = 13$$

, a Circle Centered at the origin & with radius $\sqrt{13}$.

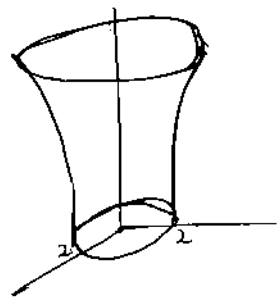
c) $Z = \sqrt{x^2+y^2-4} \Longrightarrow z^2 = x^2+y^2-4$ (5)

$$\Longrightarrow 4 = x^2+y^2-z^2$$

$$\Longrightarrow \frac{x^2}{4}+\frac{y^2}{4}-\frac{z^2}{4} = 1$$

a hyperboloid of one sheet whose axis is the z-axis.

Since $Z = \sqrt{x^2+y^2-4} \geq 0$, Then the graph of f is The upper part of the hyperboloid of one sheet $\frac{x^2}{4}+\frac{y^2}{4}-\frac{z^2}{4} = 1$, i.e the part that lies on & above The xy-plane.



$\boxed{8}$ a) $\lim\limits_{(x,y)\to(0,0)} \frac{x-2}{x^4+y^6}$ $\left(\frac{-2}{0} \rightsquigarrow \pm\infty\right)$

Numerator approaches -2

Denominator gets smaller & smaller until it approaches zero

So $\frac{\text{Numerator}}{\text{Denominator}}$ gets larger & larger without bound.

This gives $\lim\limits_{(x,y)\to(0,0)} \frac{x-2}{x^4+y^6} = +\infty$ or $-\infty$.

Is it $+\infty$ or $-\infty$? We need to check the sign of $f(x,y) = \frac{x-2}{x^4+y^6}$ near $(0,0)$. Near $(0,0)$, $x-2 < 0$ and $x^4+y^4 > 0$. So $f(x,y) < 0$. Thus

$$\lim_{(x,y)\to(0,0)} \frac{x-2}{x^4+y^6} = -\infty.$$

b) $\lim\limits_{(x,y)\to(0,0)} \frac{1-\cos(\sqrt{y})}{3x^2-y}$. $\left(\frac{0}{0}, \text{undefined}\right)$

• along the curve $C_1 : y=0$ (x-axis)

$$\lim_{\substack{(x,y)\to(0,0)\\(C_1:\ y=0)}} \frac{1-\cos\sqrt{y}}{3x^2-y} = \lim_{x\to 0} \frac{1-\cos 0}{3x^2-0} = \lim_{x\to 0} 0 = 0$$

- along the curve $C_2: x=0$ (the y-axis)

$$\lim_{\substack{(x,y)\to(0,0)\\ C_2:\, x=0}} \frac{1-\cos\sqrt{y}}{3x^2-y} = \lim_{y\to 0} \frac{1-\cos\sqrt{y}}{-y} \quad \left(\tfrac{0}{0}\right)$$

$$\overset{L.H.}{=} \lim_{y\to 0} \frac{\sin\sqrt{y}}{-2\sqrt{y}}$$

$$\overset{L.H.}{=} \lim_{y\to 0} \frac{\frac{\cos\sqrt{y}}{2\sqrt{y}}}{-\frac{1}{\sqrt{y}}} = \lim_{y\to 0} -\frac{\cos\sqrt{y}}{2} = -\frac{1}{2}$$

Since the limits along C_1 & C_2 are not equal, then the original limit does not exist.

$\boxed{5}$ $f(x,y) = \sin^{-1}x + \tan^{-1}(xy)$. $\frac{\partial^3 f}{\partial x^2 \partial y}$

- $$\frac{\partial f}{\partial y} = 0 + \frac{1}{1+(xy)^2}\cdot x = \frac{x}{1+x^2y^2}$$

- $$\frac{\partial^2 f}{\partial x \partial y} = \frac{(1+x^2y^2)\cdot 1 - x(0+2xy^2)}{(1+x^2y^2)^2} = \frac{1-x^2y^2}{(1+x^2y^2)^2}$$

- $$\frac{\partial^3 f}{\partial x^2 \partial y} = \frac{(1+x^2y^2)^2\cdot(-2xy^2) - (1-x^2y^2)\cdot 2(1+x^2y^2)(2xy^2)}{(1+x^2y^2)^4} \quad \text{Simplify}$$

$$= \frac{(1+x^2y^2)(-2xy^2)\left[(1+x^2y^2) + 2(1-x^2y^2)\right]}{(1+x^2y^2)^4}$$

$$= \frac{(-2xy^2)(3-x^2y^2)}{(1+x^2y^2)^3}.$$

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