King Fahd University of Petroleum and Minerals
Department of Mathematical Sciences
Math201.03&04, **Exam I**, Semester 052
March 15, 2006, (15 safar, 1427)
<u>Allowed Time: 90 minutes</u>

1. Convert the polar equation $r = \frac{1}{2}\tan\theta\sec\theta$ to a rectangular equation. (7 points)
2. Consider the parametric curve given by $x = t^2 + 1$, $y = t^4 + 2$, $0 \le t < +\infty$. (7 points each)
   a. Find the equation of the tangent line to the parametric curve at the point where $t = 1$.
   b. Find $\frac{d^2y}{dx^2}$.
   c. Graph the parametric curve.
3. Find the equation of the tangent line to the polar curve $r = 1 + \cos\theta$ at the point where $\theta = \frac{\pi}{3}$. (8 points)
4. Find the area of the region of intersection between the curves $r = 4\cos\theta$ and $r = 2$. (15 points)
5. (7 points each)
   a. Describe the graph of the equation $x^2 + y^2 + z^2 - 6x + z + \frac{37}{4} = 0$.
   b. Find the equation of the sphere that has center $(1,-1,-2)$ and passes through $(0,-1,-2)$.
   c. Describe, in words, how to graph the equation $z = \sqrt{y}$ in 3-space and then sketch the graph.
6. Let **u**=<4, 0, -1>, **v**=<2, 1, -2> and **w**=<1, 1, 0> be three vectors in 3-space. (7 points each)
   a. Find a vector of norm 3 having the same direction as that of **v**+**w**.
   b. If **u**- a**v** is orthogonal to **w**, then find the value of a.
   c. Find the orthogonal projection of **w** on **v**.
7. Is it possible for a vector to have the following direction cosines: $\cos\alpha = \frac{1}{3}$, $\cos\beta = \frac{2}{3}$ and $\cos\gamma = \frac{3}{4}$. Explain your answer. (7 points)

All the best,
Dr. Ibrahim Al-Rasasi

Solution of Exam I - 052

[1] $r=\frac{1}{2}\tan\theta\sec\theta \Longrightarrow r=\frac{1}{2}\,\frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\cos\theta} \Longrightarrow 2r\cos^2\theta=\sin\theta$

$\Longrightarrow 2r^2\cos^2\theta = r\sin\theta \Longrightarrow 2x^2=y$, a parabola

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[2] $x=t^2+1\ ,\ y=t^4+2,\quad 0\le t<+\infty$

a) point: $t=1 \Longrightarrow x=2\ ,\ y=3 \Longrightarrow$ point $(2,3)$

$$\text{slope}=\left.\frac{dy}{dx}\right|_{t=1}=\left.\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right|_{t=1}=\left.\frac{4t^3}{2t}\right|_{t=1}=\left.2t^2\right|_{t=1}=2$$

Equation of the tangent line is

$$y-3=2(x-2) \Longrightarrow y=2x-1$$

b) $$y'=\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{4t^3}{2t}=2t^2$$

$$\frac{d^2y}{dx^2}=\frac{dy'}{dx}=\frac{\frac{dy'}{dt}}{\frac{dx}{dt}}=\frac{4t}{2t}=2$$

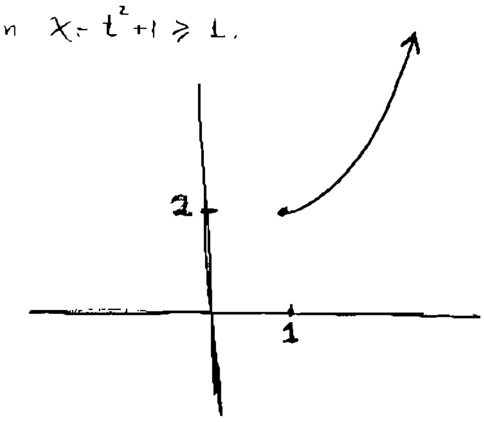
c) We eliminate the parameter t:

$x=t^2+1 \Longrightarrow t^2=x-1.$

Substitute the value of t^2 in the second equation to get

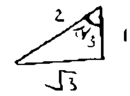
$$y=(x-1)^2+2$$

Since $t\ge 0$, then $x=t^2+1\ge 1$.



3 • Point: $x = r\cos\theta = (1+\cos\theta)\cos\theta$

$y = r\sin\theta = (1+\cos\theta)\sin\theta$



substitute $\theta = \frac{\pi}{3}$ to get $x = (1+\frac{1}{2})\cdot\frac{1}{2} = \frac{3}{4}$

$y = (1+\frac{1}{2})\cdot\frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{4}$

point $= \left(\frac{3}{4}, \frac{3\sqrt{3}}{4}\right)$

• Slope $= \left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{3}} = \dfrac{r\cos\theta + \sin\theta\cdot\frac{dr}{d\theta}}{-r\sin\theta + \cos\theta\cdot\frac{dr}{d\theta}} = \dfrac{(1+\cos\theta)\cos\theta + \sin\theta\cdot(-\sin\theta)}{-(1+\cos\theta)\sin\theta + \cos\theta\,(-\sin\theta)}$

$$= \left.\frac{\cos\theta + \cos^2\theta - \sin^2\theta}{-\sin\theta - 2\sin\theta\cos\theta}\right|_{\theta=\frac{\pi}{3}} = \frac{\frac{1}{2}+\frac{1}{4}-\frac{3}{4}}{-\frac{\sqrt{3}}{2} - 2\cdot\frac{\sqrt{3}}{2}\cdot\frac{1}{2}} = 0$$

• Equation of tangent line is

$$y - \frac{3\sqrt{3}}{4} = 0\left(x - \frac{3}{4}\right) \Rightarrow y = \frac{3\sqrt{3}}{4}.$$

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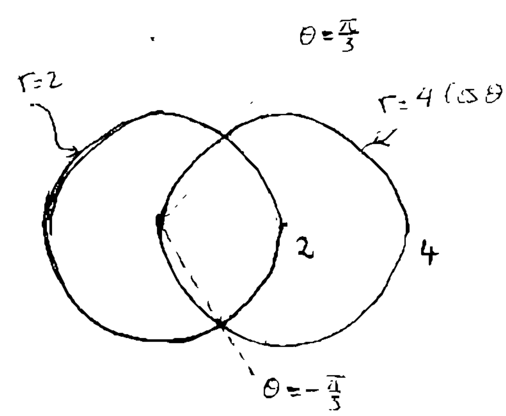
4 • Points of Intersection

$4\cos\theta = 2 \Rightarrow \cos\theta = \frac{1}{2}$

$\Rightarrow \theta = \frac{\pi}{3}, -\frac{\pi}{3}$



• By symmetry of the region, we find the area of the region in the first quadrant & then multiply by 2.

$$\bullet\ A = 2\cdot\left[\frac{1}{2}\int_0^{\pi/3}(2)^2\,d\theta + \frac{1}{2}\int_{\pi/3}^{\pi/2}(4\cos\theta)^2\,d\theta\right]$$

$$= \int_0^{\pi/3} 4\,d\theta + \int_{\frac{\pi}{3}}^{\pi/2} 8\left(1+\cos(2\theta)\right)d\theta$$

$$= \frac{4\pi}{3} + 8\left[\theta + \frac{1}{2}\sin(2\theta)\right]_{\frac{\pi}{3}}^{\pi/2}$$

$$= \frac{4\pi}{3} + 8\left[\left(\frac{\pi}{2}+\frac{1}{2}\sin\pi\right) - \left(\frac{\pi}{3}+\frac{1}{2}\sin\left(\frac{2\pi}{3}\right)\right)\right]$$

$$= \frac{4\pi}{3} + 8\left[\frac{\pi}{2} - \left(\frac{\pi}{3}+\frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right)\right]$$

$$= \frac{4\pi}{3} + 8\left[\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right]$$

$$= \frac{4\pi}{3} + \frac{4\pi}{3} - 2\sqrt{3}$$

$$= \frac{8\pi}{3} - 2\sqrt{3}$$

5

a) Complete the square in x & z:

$$x^2-6x+9+y^2+z^2+z+\frac{1}{4} = -\frac{37}{4}+\frac{1}{4}+9$$

$$\Rightarrow (x-3)^2+y^2+(z+\tfrac{1}{2})^2=0$$

The graph is a point, namely the point $(3, 0, -\frac{1}{2})$

b) Center $=(1,-1,-2)$

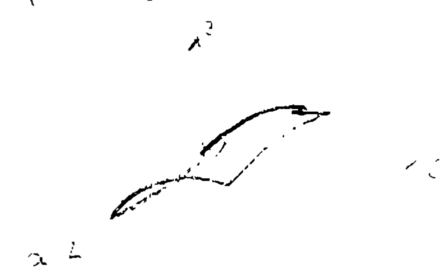
radius $= r =$ distance from $(1,-1,-2)$ to $(0,-1,-2)$

$= \sqrt{1+0+0} = 1$

Equation of the sphere is

$$(x-1)^2+(y+1)^2+(z+2)^2=1$$

c) To graph $z=\sqrt{y}$ in 3-space, first graph $z=\sqrt{y}$ in the yz-plane & then translate the graph along (parallel to) the x-axis.



6 $\vec{u}=\langle 4,0,-1\rangle$, $\vec{v}=\langle 2,1,-2\rangle$, $\vec{w}=\langle 1,1,0\rangle$

a) $\vec{v}+\vec{w}=\langle 3,2,-2\rangle$

The required vector is $3\cdot\dfrac{\vec{v}+\vec{w}}{\|\vec{v}+\vec{w}\|}$

$$=\frac{3}{\|\vec{v}+\vec{w}\|}\,\vec{v}+\vec{w}$$

$$=\frac{3}{\sqrt{9+4+4}}\langle 3,2,-2\rangle$$

$$=\frac{3}{\sqrt{17}}\langle 3,2,-2\rangle$$

$$=\left\langle \frac{9}{\sqrt{17}}, \frac{6}{\sqrt{17}}, \frac{-6}{\sqrt{17}}\right\rangle.$$

b) If $\vec{u} - a\vec{v}$ is orthogonal to $\vec{w}$, then

$$(\vec{u} - a\vec{v}) \cdot \vec{w} = 0$$

$$\Rightarrow \langle 4-2a, -a, -1+2a \rangle \cdot \langle 1, 1, 0 \rangle = 0$$

$$\Rightarrow (4-2a)(1) + (-a)(1) + (-1+2a)(0) = 0$$

$$\Rightarrow 4 - 2a - a = 0$$

$$\Rightarrow a = \frac{4}{3}$$

c) $$\text{proj}_{\vec{v}} \vec{w} = \left(\frac{\vec{w} \cdot \vec{v}}{\|\vec{v}\|^2} \right) \vec{v}$$

$$= \frac{2 + 1 + 0}{(\sqrt{4+1+4})^2} \langle 2, 1, -2 \rangle$$

$$= \frac{3}{9} \langle 2, 1, -2 \rangle$$

$$= \langle \tfrac{2}{3}, \tfrac{1}{3}, \tfrac{-2}{3} \rangle$$

$\boxed{7}$ $\cos\alpha$, $\cos\beta$ & $\cos\gamma$ have to satisfy the identity

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

Since $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = (\tfrac{1}{3})^2 + (\tfrac{2}{3})^2 + (\tfrac{3}{4})^2 = \frac{1}{9} + \frac{4}{9} + \frac{9}{16}$

$$= \frac{5}{9} + \frac{9}{16} = \frac{80 + 81}{144}$$

$$= \frac{161}{144} \neq 1,$$

then there is no vector having the given direction cosines.

Done!