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Math 201- Solved Problems

I. Polar Coordinates

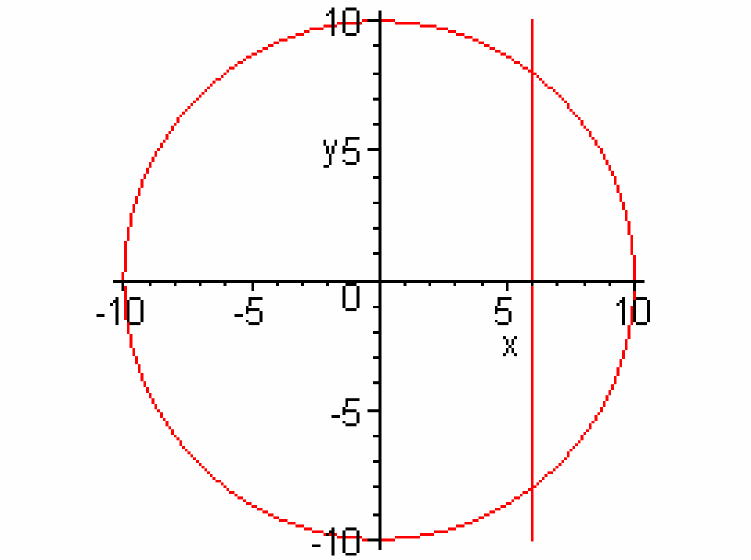
1. Describe the graph of the polar equation $r = -2$ in the polar plane.
Ans: The graph consists of all points $(-2, \theta)$ where θ is arbitrary. This means that for any angle θ, the distance from the pole to the point $(-2, \theta)$ is 2. Therefore the graph is a circle of radius 2 centered at the pole.

2. Convert the polar equation $r = 4\cos\theta$ to a Cartesian equation.
Ans: Multiply both sides of the equation by r to get $r^2 = 4r\cos\theta$ which reduces to $x^2 + y^2 = 4x$. Completing the square gives $(x-2)^2 + y^2 = 4$. A circle of radius 2 centered at (2, 0).

3. Convert the polar equation $r = \tan\theta$ to an equation in rectangular coordinates.
Ans: $\pm\sqrt{x^2 + y^2} = \frac{y}{x}$. Square both sides and then cross multiply to get $x^2(x^2 + y^2) = y^2$.

4. Find the arc length of the polar curve $r = \cos\theta$.
Ans: The graph is a circle that is traced out once as we go from $\theta = 0$ to $\theta = \pi$. (To see this, graph the curve in detail.) Thus the arc length is
$$L = \int_0^{\pi} \sqrt{(\cos\theta)^2 + (-\sin\theta)^2}\, d\theta = \int_0^{\pi} d\theta = \pi .$$

5. Find the area of the region lying inside the circle $r = 10$ and to the right of the line $r = 6\sec(\theta)$.
Ans:



The region is symmetric about the polar axis. So it is enough to find the area in the first quadrant and then multiply it by two. We first find the point of intersection in the first quadrant: $6\sec\theta = 10 \Rightarrow \cos\theta = \frac{3}{5} \Rightarrow \theta = \cos^{-1}(\frac{3}{5})$, which lies in the first quadrant by the definition the inverse cosine function.. The required area is

$$A = 2\cdot\int_0^{\cos^{-1}(3/5)} \frac{1}{2}(100 - 36\sec^2\theta)d\theta = 100\theta - 36\tan\theta]_0^{\cos^{-1}(3/5)}$$

$$= 100\cos^{-1}(3/5) - 36\cdot\frac{4}{3} = 100\cos^{-1}(3/5) - 48.$$

6. At how many points do the polar curves $r = 1+\cos\theta$ and $r = 3\cos\theta$ intersect.

Ans: They intersect at **THREE** points. Equating the equations of the two polar curves gives

$$1+\cos\theta = 3\cos\theta \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}, \frac{5\pi}{3} \in [0, 2\pi].$$

This gives two points of intersection in the interval $[0, 2\pi]$. Where is the third point? The third point is the pole as can be seen from the graphs of the two curves. We missed the pole when equating the two equations of the curves because the two curves pass through the pole at different values of θ: The cardioid passes through the pole when $\theta = \pi$ and the circle passes through the pole when $\theta = \frac{\pi}{2}$. Thus, when finding points of intersections of polar curves, it is important to graph the polar curves.

II. Three-Dimensional Space and Vectors

1. Find the equation of the sphere with center (-1,-2,-3) that is tangent to the $yz-plane$.

Ans: We need to find the radius r of the sphere. Well, r is the perpendicular distance from the center to the $yz-plane$, which is just $|-1| = 1$. The equation of the sphere is

$$(x+1)^2 + (y+2)^2 + (z+3)^2 = 1.$$

2. Prove that $\mathbf{u}\times\mathbf{v}$ is orthogonal to **u**.

Ans: Let u=$<u_1, u_2, u_3>$ and v=$<v_1, v_2, v_3>$ be two vectors in 3-space. To show that $\mathbf{u}\times\mathbf{v}$ is orthogonal to **u**, it is enough to show that their dot product is zero, i. e., $(\mathbf{u}\times\mathbf{v})\bullet\mathbf{u}$=0. First we have, by definition,

$$\mathbf{u}\times\mathbf{v} = <u_2v_3 - u_3v_2, -(u_1v_3 - u_3v_1), u_1v_2 - u_2v_1>.$$

Then, we have

$$(\mathbf{u}\times\mathbf{v})\bullet\mathbf{u} = (u_2v_3 - u_3v_2)u_1 - (u_1v_3 - u_3v_1)u_2 + (u_1v_2 - u_2v_1)u_3 = 0.$$

3. Where does the line $x = 1+t, y = 3-t, z = 2t$ intersect the $xz-plane$.

Ans: The line intersects the $xz-plane$ at the point where $y = 3 - t = 0$, that is, when $t = 3$. Substituting this value of t in the parametric equations of the line gives the point of intersection $(x, y, z) = (4,0,6)$.

4. Determine whether the line $L: x = 1+t, y = 2t, z = -1+3t$ and the plane $P: x - y + 2z = 3$ are parallel, perpendicular, or neither.

Ans: The vector parallel to the line is **v**=<1, 2, 3> and the vector normal to the plane is **n**=<1, -1, 2>. The line and the plane are parallel if and only if **v** and **n** are orthogonal and they are perpendicular if and only if **v** and **n** are parallel. Since $\mathbf{v}\times\mathbf{n}$=<7, 1, -3>$\neq \mathbf{0}$ and $\mathbf{v}\bullet\mathbf{n}=5\neq 0$, then **v** and **n** are neither parallel nor orthogonal and hence the line and the plane are neither perpendicular nor parallel.

5. Describe the curve of intersection of the paraboloids $z = x^2 + y^2$ and $z = 4 - x^2 - y^2$.

Ans: Equating $x^2 + y^2 = 4 - x^2 - y^2$, we get $x^2 + y^2 = 2$, which is a circle. Where does this circle lie? Since this circle is on the paraboloid $z = x^2 + y^2$, then the circle lies in the plane $z = 2$. We conclude that the curve of intersection of the two given paraboloids is a circle lying in the plane $z = 2$ and with center (0, 0, 2) and radius $r = \sqrt{2}$.

6. Describe the graph of the equation $\phi = 0$ in a spherical coordinate system.

Ans: The graph consists of all points P in 3-space with the property that the line segment OP makes an angle of 0 (the value of ϕ) with the positive $z-axis$. Thus, the graph is just the positive $z-axis$.

III. **Functions of Two or More Variables**.

1. Find the domain of the function $f(x,y) = \ln(\frac{x^2-1}{y})$.

Ans: We must have $\frac{x^2-1}{y}>0$.

- If $y>0$, then $x^2-1>0$ and hence $|x|>1$.
- If $y<0$, then $x^2-1<0$ and hence $|x|<1$.
- $y \neq 0, x \neq \pm 1$

So the domain, in set notation, is given by
$\{(x,y)$ in 2-space: $y>0$ and $|x|>1\} \cup \{(x,y)$ in 2-space: $y<0$ and $|x|<1\}$.

2. Find $\lim_{(x,y)\to(1,0)} \frac{y^2}{x^3+y^2-1}$.

Ans: By direct substitution, we get 0/0, which is undefined. We try to find the limits along different curves. Let C_1 : the line $x = 1$ and C_2 : the $x-axis$, $y = 0$. Then

- $\lim_{(x,y)\to(1,0)/C_1} \frac{y^2}{x^3+y^2-1} = \lim_{y\to 0} \frac{y^2}{1+y^2-1} = \lim_{y\to 0} 1 = 1$.

- $\lim_{(x,y)\to(1,0)/C_2} \dfrac{y^2}{x^3+y^2-1} = \lim_{x\to 1} \dfrac{0}{x^3+0-1} = \lim_{x\to 1} 0 = 0$.

Since the two limits are not equal, then the original limit does not exist.

3. Find $\lim_{(x,y,z)\to(0,0,0)} \dfrac{x^2+y^2-z^2}{x^2+2y^2+3z^2}$.

Ans: By direct substitution, we get 0/0, which is undefined. We try to find the limits along different curves. Let $C_1 : x = t, y = 0, z = 0$ (the $x-axis$) and $C_2 : x = 0, y = t, z = 0$ (the $y-axis$). Then

- $\lim_{(x,y,z)\to(0,0,0)/C_1} \dfrac{x^2+y^2-z^2}{x^2+2y^2+3z^2} = \lim_{t\to 0} \dfrac{t^2}{t^2} = \lim_{t\to 0} 1 = 1$.
- $\lim_{(x,y,z)\to(0,0,0)/C_2} \dfrac{x^2+y^2-z^2}{x^2+2y^2+3z^2} = \lim_{t\to 0} \dfrac{t^2}{2t^2} = \lim_{t\to 0} \dfrac{1}{2} = \dfrac{1}{2}$.

Since the two limits are not equal, then the original limit does not exist.

4. Find f_{wyx} if $f(x,y,z,w) = x^3 e^{yw} \sin^2(xz)$.

Ans: The order of differentiation is $\to$:

- $f_w(x,y,z,w) = x^3 y e^{yw} \sin^2(xz)$
- $f_{wy}(x,y,z,w) = x^3 \sin^2(xz)[ywe^{yw} + e^{yw}]$
- $f_{wyx}(x,y,z,w) = [x^3 \cdot 2\sin(xz)\cdot\cos(xz)\cdot z + 3x^2 \sin^2(xz)]e^{yw}(yw+1)$

5. Let $z^2 + \sin(xyz) = \cos x$. If z is a function of x and y, then find $\dfrac{\partial z}{\partial y}$.

Ans: We use implicit differentiation. Differentiating both sides with respect to y, we get

$$2z\frac{\partial z}{\partial y} + \cos(xyz)\cdot x[y\frac{\partial z}{\partial y} + z] = 0 \Rightarrow \frac{\partial z}{\partial y} = \frac{-xz\cos(xyz)}{2z + xy\cos(xyz)}.$$

6. Let $z = f(2+xy)$. Show that $x\dfrac{\partial z}{\partial x} - y\dfrac{\partial z}{\partial y} = 0$.

Ans: Let $u = 2 + xy$. Then we have $z = f(u), u = 2 + xy$. Using the chain rule, we get

- $\dfrac{\partial z}{\partial x} = \dfrac{dz}{du}\cdot\dfrac{\partial u}{\partial x} = \dfrac{dz}{du}\cdot y$
- $\dfrac{\partial z}{\partial y} = \dfrac{dz}{du}\cdot\dfrac{\partial u}{\partial y} = \dfrac{dz}{du}\cdot x$

This gives $x\dfrac{\partial z}{\partial x} - y\dfrac{\partial z}{\partial y} = \dfrac{dz}{du}\cdot xy - \dfrac{dz}{du}yx = 0$.

7. Locate all relative maxima, relative minima and saddle points of $f(x,y) = x^3 - 4xy + 2y^2 - 3$.

Ans: We start by finding the critical points:

$$f_x(x,y) = 3x^2 - 4y, f_y(x,y) = -4x + 4y$$

Both f_x and f_y exist everywhere. So the critical points occur when both $f_x(x,y) = 0$ and $f_y(x,y) = 0$,i.e, we need to solve the system

- $3x^2 - 4y = 0$
- $-4x + 4y = 0$

The solutions are $(0,0)$ and $(\frac{4}{3}, \frac{4}{3})$. Next we calculate

$$D = f_{xx}f_{yy} - f_{xy}{}^2 = 6x \cdot 4 - (-4)^2 = 24x - 16.$$

Since $D(0,0) = -16$<0, then f has a saddle point at (0,0). Since $D(\frac{4}{3}, \frac{4}{3}) = 16$ >0 and $f_{xx}(\frac{4}{3}, \frac{4}{3}) = 8$ >0, then f has a relative minimum at $(\frac{4}{3}, \frac{4}{3})$.