King Fahd University of Petroleum and Minerals
Department of Mathematical Sciences
Math 232, **Final Exam**, Semester 061
Sun., Jan. 21, 2007 (Muharram 2, 1428)
<u>Allowed Time: 3 hours</u>

1. (3+3+3 = 9 points)
   a. Let $I = \{0,1,2,3,...\}$. Consider the family of sets $\{A_i : i \in I\}$, where $A_i = \{1+ik : k \in Z^+\}$. Find $\bigcap_{i\in I} A_i$. Justify your answer.
   b. Give an example of a function $f : N \to N$ which is one-to-one, but not onto.
   c. Consider the group Z_7, under addition modulo 7: $+_7$. Find $ord(2)$.
2. Let A, B, C, D be nonempty sets. Prove that $(A\times B)\cup(C\times D) \subseteq (A\cup C)\times (B\cup D)$. (7 points)
3. Prove that 39 divides $2^{3n}5^n - 1$ for each positive integer $n \geq 1$. (7 points)
4. (4 +4 = 8 points)
   a. Determine whether or not the interval [0, 1] is well-ordered.
   b. Find, if possible, a nonempty well-ordered subset of real numbers that contains no integers.
5. (5+5 = 10 points)
   a. Find the remainder when 1428^{2007} is divided by 5.
   b. Prove that if $\gcd(a,b) = 1$ and $c \mid a$, then $\gcd(c,b) = 1$.
6. Let R be the set of real numbers. Define a relation ~ on $R\times R$ as follows: for $(a,b),(c,d) \in R\times R$, $(a,b) \sim (c,d)$ if and only if $a^2 + b^2 = c^2 + d^2$.
   a. Prove that ~ is an equivalence relation. (6 points)
   b. Describe the equivalence class $[(1,0)]_\sim$. (2 points)
7. Let $f : X \to Y$ be a function. Let I be an index set. Let $\{B_i : i \in I\}$ be a family of subsets of Y. (6+4 = 10 points)
   a. Prove that $f^{-1}(\bigcap_{i\in I} B_i) = \bigcap_{i\in I} f^{-1}(B_i)$.
   b. Let B be a subset of Y. Prove that $f(f^{-1}(B)) \subseteq B$.
8. (5+5 = 10 points)
   a. Prove that $[2,5) \approx (1,3]$.
   b. Let A and B be nonempty sets. Prove that if A is countable and B is uncountable, then $A\cup B$ is uncountable.
9. Define an operation $*$ on Q^+, the set of positive rational numbers, as follows: for $a,b \in Q^+$, $a*b = 3ab$. Prove that Q^+ is a group under $*$. (10 points)
10. Find all subgroups of the group Z_6 under addition modulo 6, $+_6$. (7 points)
11. Prove that if H and K are subgroups of the group G, then $H\cap K$ is a subgroup of G. (7 points)
12. Prove that the group $Q^* = Q\setminus\{0\}$, under ordinary multiplication, is not cyclic. (7 points)

All the best,
Dr. Ibrahim Al-Rasasi