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Section 7.4: Integration of Rational Functions by Partial Fractions

The main objective of this section is to integrate rational functions:

$$\text{Rational Function } = \frac{\text{a Polynomial}}{\text{a Polynomial} \neq 0}.$$

The method that will be used starts by first writing the rational function as a sum of simpler fractions (the process is called **Decomposition into Partial Fractions**) and then integrating these simpler fractions by using the previous techniques of integration.

Preliminary Remarks:

1. Equality of two polynomials: Two polynomials are equal if their corresponding coefficients are equal; that is

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0$$

if and only if $a_n = b_n, a_{n-1} = b_{n-1}, \cdots, a_1 = b_1, a_0 = b_0$.

2. A quadratic polynomial $ax^2 + bx + c$ is called **irreducible** (i.e., cannot be written as a product of two linear polynomials with real coefficients) if and only if $b^2 - 4ac < 0$. For example
   i. $x^2 + x + 1$ is irreducible since $1^2 - 4(1)(1) = -3 < 0$.
   ii. $x^2 + 3x + 2$ is reducible since $3^2 - 4(1)(2) = 1 \not< 0$. In fact, $x^2 + 3x + 2 = (x+1)(x+2)$.
3. Every polynomial can be written as a product of linear and irreducible quadratic polynomials.

1| Decomposition into Partial Fractions

Steps:

1. If deg of Num. < deg of Deno, then go to step 2. If deg of Num ≥ deg of Deno, then divide first using long division and then go to step 2.
2. Factor the denominator.
3. We proceed as follows:
   a. If the Deno contains a factor of the form $(ax+b)^n$, then this factor contributes n fractions of the form

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(ax+b)^3} + \cdots + \frac{A_n}{(ax+b)^n}$$

where $A_1, A_2, A_3, \cdots, A_n$ are constants to be found.

   b. If the Deno contains a factor of the form $(ax^2+bx+c)^n, b^2 - 4ac < 0$, then this factor contributes n fractions of the form

$$\frac{A_1 x + B_1}{ax^2+bx+c} + \frac{A_2 x + B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_n x + B_n}{(ax^2+bx+c)^n}$$

where $A_1, A_2, \cdots, A_n, B_1, B_2, \cdots, B_n$ are constant to be found.

Example 1: Find the form of the partial fraction decomposition of the fraction

$$\frac{x^3+x+17}{(2x+1)(x+4)^2(x^2+1)^3(x^2+x+1)}$$

Solution: Note first that x^2+1 and x^2+x+1 are irreducible quadratic polynomials. We get

$$\frac{x^3+x+17}{(2x+1)(x+4)^2(x^2+1)^3(x^2+x+1)}=\frac{A}{2x+1}+\frac{B}{x+4}+\frac{C}{(x+4)^2}$$
$$+\frac{Dx+E}{x^2+1}+\frac{Fx+G}{(x^2+1)^2}+\frac{Hx+I}{(x^2+1)^3}$$
$$+\frac{Jx+K}{x^2+x+1}$$

Example 2: Find the partial fraction decomposition of $\frac{x-5}{x^2-1}$.

Solution:

- deg of Num. < deg of Deno
- Deno $=x^2-1=(x-1)(x+1)$
- $\frac{x-5}{(x-1)(x+1)}=\frac{A}{x-1}+\frac{B}{x+1}$, Find A, B ?

  $=\frac{A(x+1)+B(x-1)}{(x-1)(x+1)}$

This implies that $x-5=A(x+1)+B(x-1)$.

Method I: Equating coefficients.

$$x-5=Ax+A+Bx-B=(A+B)x+(A-B)$$

This implies that

$$1=A+B$$
$$-5=A-B$$

Solving the system, we get $A=-2$ and $B=3$.

Method II: Substituting values of x (e.g., zeros of the Deno)

$x=-1$: $\quad -1-5=A(-1+1)+B(-1-1)\Rightarrow B=3$

$x=1$: $\quad 1-5=A(1+1)+B(1-1)\Rightarrow A=-2$

Thus we get the decomposition $\frac{x-5}{(x-1)(x+1)}=\frac{-2}{x-1}+\frac{3}{x+1}$.

Example 3: Find the partial fraction decomposition of $\frac{x}{(x-2)(x+1)^2}$.

Solution: deg of Num < deg of Deno; Deno is already factored. Thus we get

$$\frac{x}{(x-2)(x+1)^2}=\frac{A}{x-2}+\frac{B}{x+1}+\frac{C}{(x+1)^2};\ \text{Find } A,B,C\ ?$$

$$=\frac{A(x+1)^2+B(x-2)(x+1)+C(x-2)}{(x-2)(x+1)^2}$$

This implies that

$$x=A(x+1)^2+B(x-2)(x+1)+C(x-2)$$

Substituting values of x , we get

$x=-1$: $\quad -1=A(0)+B(0)+C(-1-2)\Rightarrow C=\frac{1}{3}$

$x=2$: $\quad 2=A(9)+B(0)+C(0)\Rightarrow A=\frac{2}{9}$

$x=0$ (arbitrary): $0=A(1)+B(-2)+C(-2)\Rightarrow B=\frac{-2}{9}$

Thus we get the decomposition

$$\frac{x}{(x-2)(x+1)^2}=\frac{2/9}{x-2}+\frac{-2/9}{x+1}+\frac{1/3}{(x+1)^2}.$$

Example 4: Find $\int\frac{2x^2-5x+2}{x^3+x}dx$.

Solution: We first decompose. deg of Num. < deg of Deno; Deno $x^3+x=x(x^2+1)$. Thus we have (note that x^2+1 is irreducible.)

$$\frac{2x^2-5x+2}{x(x^2+1)}=\frac{A}{x}+\frac{Bx+C}{x^2+1};\ \text{Find } A,B,C\ ?$$

$$=\frac{A(x^2+1)+x(Bx+C)}{x(x^2+1)}$$

This implies that

$$2x^2-5x+2=A(x^2+1)+x(Bx+C)$$

$$=(A+B)x^2+Cx+A$$

Equating coefficients we easily find $A=2, C=-5, B=0.$ Now we integrate

$$\int\frac{2x^2-5x+2}{x(x^2+1)}dx=\int\frac{2}{x}-\frac{5}{x^2+1}dx=2\ln|x|-5\tan^{-1}x+C.$$

Example 5: Find $\int\frac{2x^3-x^2+2x-2}{x^4+2x^2}dx$.

Solution: We first decompose. deg of Num. < deg of Deno; Deno $=x^2(x^2+2)$ and x^2+2 is irreducible. Thus we get

$$\frac{2x^3-x^2+2x-2}{x^2(x^2+2)}=\frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+2};\ \text{Find } A,B,C,D\ ?$$

$$=\frac{Ax(x^2+2)+B(x^2+2)+(Cx+D)x^2}{x^2(x^2+2)}$$

This implies that

$$2x^3 - x^2 + 2x - 2 = Ax(x^2+2) + B(x^2+2) + (Cx+D)x^2$$
$$= (A+C)x^3 + (B+D)x^2 + 2Ax + 2B$$

Equating coefficients we get $B = -1, A = 1, C = 1, D = 0.$ Now we integrate

$$\int \frac{2x^3 - x^2 + 2x - 2}{x^4 + 2x^2} dx = \int \frac{1}{x} - \frac{1}{x^2} + \frac{x}{x^2+2} dx = \ln|x| + \frac{1}{x} + \frac{1}{2}\ln(x^2+2) + C.$$

Example 6: Find $\int \frac{x^3 + x + 2}{x^2 + 2x - 8} dx$.

Solution: First we decompose. Since deg of Num. $\not<$ deg of Deno, we first divide using long Division. This gives the following

$$\frac{x^3 + x + 2}{x^2 + 2x - 8} = x - 2 + \frac{13x - 14}{x^2 + 2x - 8}$$

Next we decompose the new fraction:

$$\frac{13x - 14}{x^2 + 2x - 8} = \frac{13x - 14}{(x+4)(x-2)} = \frac{A}{x+4} + \frac{B}{x-2}.$$

We leave it to the reader to check that $A = 11$ and $B = 2.$ Now we integrate

$$\int \frac{x^3 + x + 2}{x^2 + 2x - 8} dx = \int x - 2 + \frac{11}{x+4} + \frac{2}{x-2} dx = \frac{1}{2}x^2 - 2x + 11\ln|x+4| + 2\ln|x-2| + C.$$

<u>A Basic Formula:</u> For $a > 0,$ we have

$$\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + C.$$

Proof: Decompose first:

$$\frac{1}{x^2 - a^2} = \frac{1}{(x-a)(x+a)} = \frac{A}{x-a} + \frac{B}{x+a}.$$

Now it is easy to show that $A = \frac{1}{2a}$ and $B = \frac{-1}{2a}.$ Now integrate and combine your result using the properties of the logarithm.

Example 7: Find $\int \frac{\cos x}{\sin^2 x - 3} dx$.

Solution: Let $y = \sin x$. Then $dy = \cos x \; dx$. Thus we get using the above basic formula

$$\int \frac{\cos x}{\sin^2 x - 3} dx = \int \frac{1}{y^2 - 3} dy = \frac{1}{2\sqrt{3}} \ln\left|\frac{y - \sqrt{3}}{y + \sqrt{3}}\right| + C = \frac{1}{2\sqrt{3}} \ln\left|\frac{\sin x - \sqrt{3}}{\sin x + \sqrt{3}}\right| + C.$$

2| **<u>Rationalizing Substitution</u>** (Power Substitution; Mixed Radicals; Radical Expressions)

<u>General Principle:</u> Try to remove the radicals by making a suitable power substitution (e.g., $x = u^n$) to convert the integral to an integral of a rational function.

Example 8: Find $\int \frac{\sqrt{x}}{1+\sqrt[3]{x}}dx$.

Solution: To remove both radicals, let $x = u^6$ (or $u = \sqrt[6]{x}$; note that $6 = 2\times 3$). Then $dx = 6u^5 du; \sqrt{x} = u^3; \sqrt[3]{x} = u^2$. This gives

$$\int \frac{\sqrt{x}}{1+\sqrt[3]{x}}dx = \int \frac{u^3}{1+u^2}\cdot 6u^5 du = 6\int \frac{u^8}{u^2+1}du$$

$$= 6\int (u^6 - u^4 + u^2 - 1 + \frac{1}{u^2+1})du \quad \text{(by long division)}$$

$$= 6\left[\frac{1}{7}u^7 - \frac{1}{5}u^5 + \frac{1}{3}u^3 - u + \tan^{-1}u\right] + C$$

$$= 6\left[\frac{1}{7}\sqrt[6]{x^7} - \frac{1}{5}\sqrt[6]{x^5} + \frac{1}{3}\sqrt[6]{x^3} - \sqrt[6]{x} + \tan^{-1}\sqrt[6]{x}\right] + C$$

Example 9: Find $\int \frac{1+\sqrt{x}}{1-\sqrt{x}}dx$.

Solution: To remove the radicals, we let $x = u^2$ (or $u = \sqrt{x}$). Then $dx = 2udu$ and $\sqrt{x} = u$. Thus we get

$$\int \frac{1+\sqrt{x}}{1-\sqrt{x}}dx = \int \frac{1+u}{1-u}\cdot 2udu = 2\int \frac{u^2+u}{-u+1}du = -2\int \frac{u^2+u}{u-1}du$$

$$= -2\int (u + 2 + \frac{2}{u-1})du \quad \text{(by long division)}$$

$$= -2\left[\frac{1}{2}u^2 + 2u + 2\ln|u-1|\right] + C$$

$$= -x - 4\sqrt{x} - 4\ln\left|\sqrt{x} - 1\right| + C.$$

3| Integrating Rational Expressions of $\sin x$ and $\cos x$

Here we will consider integrals of the form

$$\int \frac{1}{3+\sin x}dx; \ \int \frac{\cos^2 x}{\cos x - \sin^3 x}dx; \ \int \frac{1}{\sin x + \tan x}dx.$$

These can be integrated by making the following substitution:

Let $t = \tan\left(\frac{x}{2}\right), \ -\frac{\pi}{2} < \frac{x}{2} < \frac{\pi}{2}$. Then

$$dx = \frac{2}{1+t^2}dt; \ \sin x = \frac{2t}{1+t^2}; \ \cos x = \frac{1-t^2}{1+t^2}.$$

See the details in class.

Example 10: Find $\int \frac{1}{3+\sin x}dx$.

Solution: Let $t = \tan\left(\frac{x}{2}\right), \ -\frac{\pi}{2} < \frac{x}{2} < \frac{\pi}{2}$. Then

$$dx = \frac{2}{1+t^2}dt \text{ and } \sin x = \frac{2t}{1+t^2}.$$

Thus we have

$$\int \frac{1}{3+\sin x}dx = \int \frac{1}{3+\frac{2t}{1+t^2}} \cdot \frac{2}{1+t^2}dt$$

$$= \int \frac{2}{3(1+t^2)+2t}dt$$

$$= 2\int \frac{1}{3t^2+2t+3}dt; \text{ Deno is irred. quadratic; complete the aquare}$$

$$= \frac{2}{3}\int \frac{1}{(t+\frac{1}{3})^2+\frac{8}{9}}dt; \text{ Let } y = t+\frac{1}{3}. \text{ Then } dy = dt.$$

$$= \frac{2}{3}\int \frac{1}{y^2+(8/9)}dy$$

$$= \frac{2}{3}\cdot\frac{3}{\sqrt{8}}\tan^{-1}\left(\frac{y}{(\sqrt{8}/3)}\right)+C$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{3}{\sqrt{8}}(t+\frac{1}{3})\right)+C$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{3}{\sqrt{8}}\tan\left(\frac{x}{2}\right)+\frac{1}{\sqrt{8}}\right)+C$$