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Section: 7.3: <u>Trigonometric Substitutions</u>

In this section we consider integrals containing expressions of the forms

$$\sqrt{a^2-x^2},\ \sqrt{a^2+x^2},\ \sqrt{x^2-a^2},\ a>0.$$

To deal with such integrals, we make the following trigonometric substitutions.

Integral contains	Trig. Substitution
$\sqrt{a^2-x^2}$	$x=a\sin\theta,\quad -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$
$\sqrt{a^2+x^2}$	$x=a\tan\theta,\quad -\frac{\pi}{2}<\theta<\frac{\pi}{2}$
$\sqrt{x^2-a^2}$ Domain: $x\ge a$ or $x\le -a$	$x=a\sec\theta,\quad 0\le\theta<\pi/2\quad (x\ge a)$ $\pi\le\theta<3\pi/2\ (x\le -a)$

<u>Remarks:</u>

1. These trigonometric substitutions will remove the radicals.
2. The restriction on θ is made
   a. to easily remove the absolute values (during simplification)
   b. to make the trigonometric functions one-to-one and hence to write the substitutions in the inverse form: $\theta=\sin^{-1}(x/a)$, $\theta=\tan^{-1}(x/a)$, $\theta=\sec^{-1}(x/a)$.
3. The same trigonometric substitutions can be used if the square root $\sqrt{\cdot}$ is replaced by the powers $(\cdot)^{3/2},(\cdot)^{5/2},\cdots$.

Example 1: Find $\int\frac{x^5}{\sqrt{x^2+1}}dx$.

Solution: Let $x=\tan\theta$, $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$. Then

- $dx=\sec^2\theta\, d\theta$
- $\sqrt{x^2+1}=\sqrt{\tan^2\theta+1}=\sqrt{\sec^2\theta}=|\sec\theta|=\sec\theta$, (as $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$)

Thus we get

$$\begin{aligned}\int\frac{x^5}{\sqrt{x^2+1}}dx &=\int\frac{\tan^5\theta}{\sec\theta}\cdot\sec^2\theta\, d\theta\\ &=\int\tan^5\theta\cdot\sec\theta\, d\theta\\ &=\int\tan^4\theta\cdot\sec\theta\tan\theta\, d\theta=\int(\tan^2\theta)^2\cdot\sec\theta\tan\theta\, d\theta\\ &=\int(\sec^2\theta-1)^2\cdot\sec\theta\tan\theta\, d\theta\end{aligned}$$

Let $u=\sec\theta$. Then $du=\sec\theta\tan\theta\, d\theta$. So the last integral becomes

$$= \int (u^2-1)^2 du = \int (u^4 - 2u^2 + 1) du$$
$$= \frac{1}{5}u^5 - \frac{2}{3}u^3 + u + C$$
$$= \frac{1}{5}\sec^5\theta - \frac{2}{3}\sec^3\theta + \sec\theta + C$$
$$= \frac{1}{5}(\sqrt{x^2+1})^5 - \frac{2}{3}(\sqrt{x^2+1})^3 + \sqrt{x^2+1} + C.$$

Example 2: Find $\int \sqrt{4-x^2}dx$.

Solution: Let $x = 2\sin\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then

- $dx = 2\cos\theta\ d\theta$
- $\sqrt{4-x^2} = \sqrt{4-4\sin^2\theta} = \sqrt{4(1-\sin^2\theta)} = \sqrt{4\cos^2\theta} = 2|\cos\theta| = 2\cos\theta$

Thus we get

$$\int \sqrt{4-x^2}dx = \int 2\cos\theta \cdot 2\cos\theta\ d\theta$$
$$= 4\int \cos^2\theta\ d\theta = 4\int \frac{1+\cos(2\theta)}{2} d\theta$$
$$= 2[\theta + \frac{1}{2}\sin(2\theta)] + C$$
$$= 2\theta + 2\sin\theta\cos\theta + C$$
$$= 2\sin^{-1}(\frac{x}{2}) + 2\cdot\frac{x}{2}\cdot\frac{\sqrt{4-x^2}}{2} + C$$
$$= 2\sin^{-1}(\frac{x}{2}) + \frac{1}{2}x\sqrt{4-x^2} + C$$

Example 3: Find $\int_{\sqrt{2}}^{2} \frac{1}{x^3\sqrt{x^2-1}} dx$.

Solution: Let $x = \sec\theta,\ 0 \le \theta < \pi/2$ as the interval is positive. Then

- $dx = \sec\theta\tan\theta\ d\theta$
- $\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \sqrt{\tan^2\theta} = |\tan\theta| = \tan\theta$
- $x = \sqrt{2} \Rightarrow \sec\theta = \sqrt{2} \Rightarrow \cos\theta = 1/\sqrt{2} \Rightarrow \theta = \pi/4$
- $x = 2 \Rightarrow \sec\theta = 2 \Rightarrow \cos\theta = 1/2 \Rightarrow \theta = \pi/3$

Thus we get

$$\int_{\sqrt{2}}^{2} \frac{1}{x^3\sqrt{x^2-1}} dx = \int_{\pi/4}^{\pi/3} \frac{1}{\sec^3\theta\cdot\tan\theta}\sec\theta\tan\theta\ d\theta$$
$$= \int_{\pi/4}^{\pi/3} \cos^2\theta\ d\theta = \frac{1}{2}\int_{\pi/4}^{\pi/3} 1 + \cos(2\theta)\ d\theta$$
$$= \frac{1}{2}[\theta + \frac{1}{2}\sin(2\theta)]|_{\pi/4}^{\pi/3} = \frac{\pi + 3\sqrt{3} - 6}{24}$$

Example 4: Find $\int e^t\sqrt{4-e^{2t}}\,dt$.

Solution: Let $y=e^t$. Then $dy=e^t dt$. Thus

$\int e^t\sqrt{4-e^{2t}}\,dt=\int\sqrt{4-y^2}\,dy$. Now let $y=2\sin\theta,\ -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$ and proceed as in Example 1. The final answer is

$$\int e^t\sqrt{4-e^{2t}}\,dt=2\sin^{-1}\left(\frac{e^t}{2}\right)+\frac{1}{2}e^t\sqrt{4-e^{2t}}+C.$$

Example 5: Find $\int\sqrt{1+9x^2}\,dx$.

Solution: Let $y=3x$. Then $dy=3dx$. Thus

$\int\sqrt{1+9x^2}\,dx=\frac{1}{3}\int\sqrt{1+y^2}\,dy$. Now let $y=\tan\theta,\ -\frac{\pi}{2}<\theta<\frac{\pi}{2}$ and proceed in the usual way. The final answer is

$$\int\sqrt{1+9x^2}\,dx=\frac{3}{2}x\sqrt{1+9x^2}+\frac{1}{2}\ln\left|3x+\sqrt{1+9x^2}\right|+C.$$

Example 6: Find $\int\frac{1}{\sqrt{6x-x^2}}\,dx$.

Solution: Here we can not use trigonometric substitutions directly. First we have to complete the square of the radicand:

$$6x-x^2=-(x^2-6x)=-(x^2-6x+9-9)=-[(x-3)^2-9]=9-(x-3)^2$$

This gives

$\int\frac{1}{\sqrt{6x-x^2}}\,dx=\int\frac{1}{\sqrt{9-(x-3)^2}}\,dx$; let $y=x-3$. Then $dy=dx$.

$$=\int\frac{1}{\sqrt{9-y^2}}\,dy$$

$$=\sin^{-1}\left(\frac{y}{3}\right)+C=\sin^{-1}\left(\frac{x-3}{3}\right)+C$$