Section 7.2: Trigonometric Integrals

1 Integrating Powers of Trigonometric Functions

1.1 First Powers

Theorem:

1. $\int (\cos x)dx = \sin x + C$.
2. $\int (\sin x)dx = -\cos x + C$.
3. $\int (\tan x)dx = \ln|\sec x| + C$.
4. $\int (\cot x)dx = \ln|\sin x| + C$.
5. $\int (\sec x)dx = \ln|\sec x + \tan x| + C$.
6. $\int (\csc x)dx = \ln|\csc x - \cot x| + C$.

Proof of (5): We do a trick:

$$\int (\sec x)dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x}dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}dx.$$

Now let $u = \sec x + \tan x$. Then $du = (\sec x \tan x + \sec^2 x)dx$. Thus the last integral equals

$$\int \frac{1}{u}du = \ln|u| + C = \ln|\sec x + \tan x| + C.$$

1.2 Second Powers

Theorem:

1. $\int (\cos^2 x)dx = \frac{1}{2}\int [1+\cos(2x)]dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C$.
2. $\int (\sin^2 x)dx = \frac{1}{2}\int [1-\cos(2x)]dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$.
3. $\int (\tan^2 x)dx = \int (\sec^2 x - 1)dx = \tan x - x + C$.
4. $\int (\cot^2 x)dx = \int (\csc^2 x - 1)dx = -\cot x - x + C$.
5. $\int (\sec^2 x)dx = \tan x + C$.
6. $\int (\csc^2 x)dx = -\cot x + C$.

1.3 Higher Powers

In this case, we use basic trigonometric identities and the previous integration techniques. We illustrate the general method by giving several examples.

Examples: Find each of the following integrals.

1. $\int (\sin^3 x)dx$.

Solution: Introduce $\cos x$ in the integrand:

$$\int (\sin^3 x)dx = \int \sin^2 x \cdot \sin x \cdot dx = \int (1-\cos^2 x)\sin x \cdot dx.$$

Let $u = \cos x$. Then $du = (-\sin x)dx$. Thus the last integral becomes

$$\int (u^2 - 1)du = \frac{1}{3}u^3 - u + C = \frac{1}{3}\cos^3 x - \cos x + C.$$

2. $\int (\sin^4 x)dx$.

Solution: Use the identity: $\sin^2 x = \dfrac{1-\cos(2x)}{2}$.

$$\int (\sin^2 x)^2 dx = \int \left(\frac{1-\cos(2x)}{2}\right)^2 dx = \frac{1}{4}\int (1 - 2\cos(2x) + \cos^2(2x))dx =$$

$$\frac{1}{4}\int [1-\cos(2x) + \frac{1+\cos(4x)}{2}]dx = \frac{1}{4}\int [\frac{3}{2} - \cos(2x) + \frac{1}{2}\cos(4x)]dx =$$

$$\frac{3}{8}x - \frac{1}{8}\sin(2x) + \frac{1}{32}\sin(4x) + C.$$

3. $\int (\tan^3 x)dx$.

Solution: We introduce $\sec x$ in the integral:

$$\int \tan^2 x \cdot \tan x \cdot dx = \int (\sec^2 x - 1)\tan x \cdot dx = \int [\sec^2 x \ \tan x]dx - \int (\tan x)dx$$

The firs integral can be solved using the u-Substitution $u = \tan x$ and the second integral is immediate. The value of the integral is $\frac{1}{2}\tan^2 x - \ln|\sec x| + C$.

4. $\int (\tan^4 x)dx$.

Solution: We proceed as in Example 3 above:

$$\int \tan^2 x \cdot \tan^2 x \cdot dx = \int (\sec^2 x - 1)\tan^2 x \cdot dx = \int [\sec^2 x \ \tan^2 x]dx - \int (\tan^2 x)dx$$

The first integral can be solved using the u-Substitution $u = \tan x$ and the second integral was discussed in Subsection $\boxed{1.2}$. The value of the integral is $\frac{1}{3}\tan^3 x - \tan x + x + C$.

5. $\int (\tan^5 x)dx$.

Solution: Introducing $\sec x$, we get

$$\int \tan^4 x \cdot \tan x \cdot dx = \int (\sec^2 x - 1)^2 \tan x \cdot dx = \int (\sec^4 x - 2\sec^2 x + 1)\tan x \cdot dx =$$
$$\int (\sec^4 x \tan x)dx - 2\int (\sec^2 x \tan x)dx + \int (\tan x)dx =$$
$$\int (\sec^3 x \sec x \tan x)dx - 2\int (\sec^2 x \tan x)dx + \int (\tan x)dx$$

For the first integral, make the u-Substitution $u = \sec x$ and for the second integral make the u-Substitution $u = \tan x$. The value of the integral is $\frac{1}{4}\sec^4 x - \tan^2 x + \ln|\sec x| + C$.

2| Integrating Products of Sines and Cosines

Let m and n be positive integers.

$\int (\sin^m x \cos^n x)dx$	Steps
n is odd	• Split off a factor of $\cos x$ • Apply the identity $\cos^2 x = 1 - \sin^2 x$ • Make the u-substitution $u = \sin x$
m is odd	• Split off a factor of $\sin x$ • Apply the identity $\sin^2 x = 1 - \cos^2 x$ • Make the u-Substitution $u = \cos x$
m and n are even	Use $\cos^2 x = \frac{1+\cos(2x)}{2}$ and $\sin^2 x = \frac{1-\cos(2x)}{2}$ to reduce to an integral of powers of $\cos(2x)$

Examples: Find the following integrals.

1. $\int (\sin^3 x \cos^6 x)dx$.

Solution:

$$\int [\sin^2 x \cos^6 x]\sin x \cdot dx = \int [(1-\cos^2 x)\cos^6 x]\sin x \cdot dx = \int [\cos^6 x - \cos^8 x]\sin x \cdot dx$$

Let $u = \cos x$. Then $du = (-\sin x)dx$ and the last integral becomes

$$\int [u^8 - u^6]du = \frac{1}{9}u^9 - \frac{1}{7}u^7 + C = \frac{1}{9}\cos^9 x - \frac{1}{7}\cos^7 x + C.$$

2. $\int (\sin^2 x \cos^4 x)dx$.

Solution:

$$\int (\sin^2 x \cos^4 x)dx = \int \left(\frac{1-\cos(2x)}{2}\right)\left(\frac{1+\cos(2x)}{2}\right)^2 dx = \frac{1}{8}\int (1-\cos(2x))(1+2\cos(2x)+\cos^2(2x))dx$$

$$= \frac{1}{8}\int (1+\cos(2x) - \cos^2(2x) - \cos^3(2x))dx.$$

The first two terms are easy to integrate. The third term can be integrated by using the identity $\cos^2(2x) = \frac{1+\cos(4x)}{2}$. The third term can be integrated in exactly the same way of Example 1 in Subsection $\boxed{1.3}$. The details are left for the reader.

Next we consider integrals of the form

$$\int \sin(nx)\cos(mx)dx, \int \sin(nx)\sin(mx)dx, \int \cos(nx)\cos(mx)dx.$$

These can be integrated using the following identities:

- $\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha-\beta)+\sin(\alpha+\beta)]$
- $\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$
- $\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$

Example: Find $\int \sin(5x)\sin(7x)dx$.

Solution: Since

$$\sin(5x)\sin(7x) = \frac{1}{2}[\cos(5x-7x)-\cos(5x+7x)] = \frac{1}{2}[\cos(-2x)-\cos(12x)] = \frac{1}{2}[\cos(2x)-\cos(12x)],$$

then

$$\int \sin(5x)\sin(7x)dx = \frac{1}{2}\int[\cos(2x)-\cos(12x)]dx = \frac{1}{4}\sin(2x)-\frac{1}{24}\sin(12x)+C.$$

3| Integrating Products of Tangents and Secants

Let m and n be positive integers.

$\int(\tan^m x \sec^n x)dx$	Steps
n is even	• Split off a factor of $\sec^2 x$ • Use the identity $\sec^2 x = 1+\tan^2 x$ • Make the u-Substitution $u = \tan x$
m is odd	• Split off a factor of $\sec x \tan x$ • Use the identity $\tan^2 x = \sec^2 x - 1$ • Make the u-Substitution $u = \sec x$
m is even and n is odd	Use the identity $\tan^2 x = \sec^2 x - 1$ to reduce to an integral of powers of $\sec x$

Note: Similar methods for $\int(\cot^m x \csc^n x)dx$ are used. Use the identity $1+\cot^2 x = \csc^2 x$ suitably.

Example: Find $\int \tan^3(4x)\sec^4(4x)dx$.

Solution: $\int \tan^3(4x)\sec^4(4x)dx = \int \tan^3(4x)\sec^2(4x)\cdot\sec^2(4x)dx =$

$\int \tan^3(4x)[\tan^2(4x)+1]\sec^2(4x)dx = \int [\tan^5(4x)+\tan^3(4x)]\sec^2(4x)dx$

Let $u = \tan(4x)$. Then $du = 4\sec^2(4x)dx$. The last integral equals

$\frac{1}{4}\int (u^5+u^3)du = \frac{1}{4}[\frac{u^6}{6}+\frac{u^4}{4}] = \frac{1}{24}\tan^6(4x)+\frac{1}{16}\tan^4(4x)+C$.

Another Method:

$\int \tan^3(4x)\sec^4(4x)dx = \int \tan^2(4x)\sec^3(4x)\cdot\sec(4x)\tan(4x)dx =$

$\int [\sec^2(4x)-1]\sec^3(4x)\cdot\sec(4x)\tan(4x)dx = \int [\sec^5(4x)-\sec^3(4x)]\cdot\sec(4x)\tan(4x)dx$

Let $u = \sec(4x)$. Then $du = 4\sec(4x)\tan(4x)dx$. The last integral equals

$\frac{1}{4}\int [u^5-u^3]du = \frac{1}{4}[\frac{u^6}{6}-\frac{u^4}{4}]+C = \frac{1}{24}\sec^6(4x)-\frac{1}{16}\sec^4(4x)+C$.