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Section 11.7: Strategy for Testing Series

We start by listing the basic tests for convergence and divergence of series:

1. Telescoping series
2. Geometric series
3. The Divergence Test
4. The Integral Test
5. p-series
6. The Comparison Test
7. The Limit Comparison Test
8. The Alternating Series Test
9. The Absolute Convergence Test
10. The Ratio Test
11. The nth Root Test

Examples: Converge or Diverge?

1. $\sum_{n=1}^{+\infty} \sin n.$
2. $\sum_{n=1}^{+\infty} \tan(1/n).$
3. $\sum_{n=1}^{+\infty} \frac{\tan^{-1}(n)}{n\sqrt{n}}.$
4. $\sum_{n=0}^{+\infty} \frac{n!}{2\cdot 5\cdot 8\cdots(3n+2)}.$
5. $\sum_{n=1}^{+\infty} \frac{1}{n+n\cos^2 n}.$
6. $\sum_{n=1}^{+\infty} \frac{e^{1/n}}{n^2}.$
7. $\sum_{n=1}^{+\infty} (\sqrt[n]{2}-1).$

Solutions:

1. Use the Divergence Test. Since $\lim_{n\to+\infty} \sin n$ does not exist, then the series diverges.

2. Use LCT with $b_n = \frac{1}{n}$. Since $\lim_{n\to+\infty} \frac{\tan(1/n)}{1/n} = 1 > 0$ (use L'Hospital's Rule) and $\sum_{n=1}^{+\infty} 1/n$ diverges (the Harmonic series), then the given series diverges.

3. Use the Comparison Test $\left(\frac{\tan^{-1} n}{n\sqrt{n}} \le \frac{\pi}{2n\sqrt{n}}\right)$ or the LCT with $b_n = \frac{1}{n\sqrt{n}}$. The given series converges.

4. Use the Ratio Test. Be careful.
$a_n = \frac{n!}{2\cdot 5\cdot 8\cdots(3n+2)}$ and $a_{n+1} = \frac{(n+1)!}{2\cdot 5\cdot 8\cdots(3n+2)\cdot(3(n+1)+2)} = \frac{(n+1)\cdot n!}{2\cdot 5\cdot 8\cdots(3n+2)\cdot(3n+5)}$. (Notice how we write the denominator of a_{n+1}. This is the place where you have to be very careful.) As $\lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{3} < 1$, the given series converges.

5. Use the Comparison Test. As $\cos^2 n \le 1$, then $n\cos^2 n \le n$ and hence $n + n\cos^2 n \le 2n$. This implies that $\frac{1}{n + n\cos^2 n} \ge \frac{1}{2n}$. Since $\sum_{n=1}^{+\infty} \frac{1}{2n} = \frac{1}{2}\sum_{n=1}^{+\infty} \frac{1}{n}$ diverges (a constant times the Harmonic series), then the given series diverges by the comparison test.

6. Use the Comparison Test. As $e^{1/n} = \sqrt[n]{e} \le e$, then $\frac{e^{1/n}}{n^2} \le \frac{e}{n^2}$. Since $\sum_{n=1}^{+\infty} \frac{e}{n^2} = e\sum_{n=1}^{+\infty} \frac{1}{n^2}$ converges (a constant times a convergent p-series: $p = 2 > 1$), then the given series converges by the comparison test.

7. Use the LCT with $b_n = \frac{1}{n}$. Now $\lim_{n\to+\infty} \frac{2^{1/n} - 1}{1/n} = \ln 2$, finite and positive (using L'Hospital's Rule). Thus, since $\sum_{n=1}^{+\infty} \frac{1}{n}$ diverges (the Harmonic series) then the given series diverges by the LCT.