Dr. Ibrahim Al-Rasasi

Math 102- Solved Problems
(28 Problems, 6 pages)

1. Write the sum $\sum_{n=1}^{20} \frac{\sin(n-5)}{2n+3}$ in a sigma notation that starts with 3 rather than 1. **Solution:** Add 2 to both sides of $n=1$ to get $n+2=3$. Let $j=n+2$. Then $n=j-2$. Further, $n=1 \Rightarrow j=3$ and $n=20 \Rightarrow j=22$. Thus we have $\sum_{n=1}^{20} \frac{\sin(n-5)}{2n+3} = \sum_{j=3}^{22} \frac{\sin(j-7)}{2j-1}$.
2. Prove that $6\sqrt{2} \le \int_0^3 \sqrt{x^3+8}\ dx \le 3\sqrt{35}$. **Solution:** Here $f(x)=\sqrt{x^3+8}$ is defined over the interval [0, 3]. We first find two numbers m and M such that $m \le f(x) \le M$ and then integrate each side. Since $x \in [0, 3]$, then $0 \le x \le 3$. Cubing each side, we get $0 \le x^3 \le 27$. Adding 8 to each side, we obtain $8 \le x^3+8 \le 35$. Taking the square root of each side, we get $2\sqrt{2} \le \sqrt{x^3+8} \le \sqrt{35}$. Now we integrate over the interval [0, 3] to obtain $\int_0^3 2\sqrt{2}\ dx \le \int_0^3 \sqrt{x^3+8}\ dx \le \int_0^3 \sqrt{35}\ dx$. Evaluating, we obtain what we want: $6\sqrt{2} \le \int_0^3 \sqrt{x^3+8}\ dx \le 3\sqrt{35}$.
3. Evaluate the limit $\lim_{n\to+\infty} \sum_{i=1}^{n} \frac{1}{n}\sin(\frac{i}{n})$. **Solution:** We compare with the formula $\lim_{n\to+\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)dx.$ We observe that $\Delta x = \frac{1}{n}$ and $x_i^* = \frac{i}{n}$; that is, x_i^* is chosen to be the right endpoint of each subinterval. Since $x_i^* = a + i\Delta x$, then $\frac{i}{n} = a + i\frac{1}{n}$ and hence $a=0.$ Also, since $\Delta x = \frac{b-a}{n}$, then $\frac{1}{n} = \frac{b-0}{n}$ and hence $b=1$. Further, $f(x)=\sin x$. So the given limit equals $\int_0^1 \sin x\ dx = 1-\cos 1.$
4. Find the critical points of the function $F(x) = \int_2^{x^2-2x} e^{t^2} dt$. **Solution:** We find the derivative: $F'(x) = e^{(x^2-2x)^2}(2x-2) = 0 \Rightarrow x=1$. So the function has only one critical point: $x=1$.
5. Find $\int \cos^2 x\ dx$. **Solution:** Using the identity $\cos(2x) = 2\cos^2 x - 1$, we have $\int \cos^2 x\ dx = \int \frac{1+\cos(2x)}{2} dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C.$

6. Find $\int \frac{2}{\sqrt{x+2}-\sqrt{x}}dx.$ **Solution:** Multiplying the numerator and denominator by the conjugate $\sqrt{x+2}+\sqrt{x}$, the integral becomes $\int (\sqrt{x+2}+\sqrt{x})dx =$ $\frac{2}{3}[(x+2)^{3/2}+x^{3/2}]+C.$

7. Find $\int \left(\frac{x^2}{x+1}\right)^2 dx.$ **Solution:** First divide x^2 by $x+1$ (use long division) to get $\frac{x^2}{x+1}=x-1+\frac{1}{x+1}$. Squaring, we get $\left(\frac{x^2}{x+1}\right)^2 = (x-1)^2+\frac{2x-2}{x+1}+$ $\frac{1}{(x+1)^2}$. It easy to integrate the first and the third terms. For the second term, divide first: $\frac{2x-2}{x+1}=2-\frac{4}{x+1}$. Thus, $\int \left(\frac{x^2}{x+1}\right)^2 dx = \frac{(x-1)^3}{3}+2x-4\ln|x+1|-\frac{1}{x+1}+C.$

8. Find $\int \sqrt{x}\tan(3+x^{3/2})dx.$ **Solution:** Let $u=3+x^{3/2}$. Then $du=\frac{3}{2}x^{1/2}dx.$ We get

$$\int \sqrt{x}\tan(3+x^{3/2})dx = \frac{2}{3}\int \tan u\ du = \frac{2}{3}\ln\left|\sec u\right|+C = \frac{2}{3}\ln\left|\sec(3+x^{3/2})\right|+C\ .$$

9. Find $\int \frac{2x^3}{(x^2+1)^4}dx$. **Solution:** Let $u=x^2+1$. Then $du=(2x)dx$. We obtain

$$\int \frac{2x^3}{(x^2+1)^4}dx = \int \frac{x^2}{(x^2+1)^4}(2x)dx = \int \frac{u-1}{u^4}du = \int (u^{-3}-u^{-4})du =$$

$$\frac{u^{-2}}{-2}-\frac{u^{-3}}{-3}+C = -\frac{1}{2(x^2+1)^2}+\frac{1}{3(x^2+1)^3}+C.$$

10. Consider the function $f(x)=\frac{1}{x}$ over the interval [1, 6]. Find the values of c that satisfy the Mean Value Theorem for Integrals. **Solution:** since f is continuous on the interval [1, 6], then, by MVTI, there is a number c in [1, 6] such that $\int_1^6 f(x)dx = f(c)(6-1).$ This gives $\ln 6=\frac{5}{c}$ and hence $c=\frac{5}{\ln 6}$. Note that $\frac{5}{\ln 6}\in[1,6]$ since $6<e^5<6^6 \Rightarrow \ln 6<5<6\ln 6 \Rightarrow 1<\frac{5}{\ln 6}<6.$

11. Find $\int x\ln(x^2+1)dx.$ **Solution:** Let $y=x^2+1$. Then $dy=(2x)dx.$ Thus we get $\int x\ln(x^2+1)dx = \frac{1}{2}\int \ln y\ dy = \frac{1}{2}[y\ln y - y]+C = \frac{1}{2}y[\ln y-1]+C$ $=\frac{1}{2}(x^2+1)[\ln(x^2+1)-1]+C.$

12. Find $\int \csc^6(2x)dx.$ **Solution:** $\int \csc^6(2x)dx = \int \csc^4(2x)\csc^2(2x)dx$ $= \int (1+\cot^2(2x))^2 \csc^2(2x)dx = \int (1+2\cot^2(2x)+\cot^4(2x))\csc^2(2x)dx.$ let $u = \cot(2x).$ Then $du = -2\csc^2(2x)dx.$ The last integral equals $\frac{-1}{2}\int (1+2u^2+u^4)du = \frac{-1}{2}(u+\frac{2}{3}u^3+\frac{1}{5}u^5)+C = \frac{-1}{2}\cot(2x) - \frac{1}{3}\cot^3(2x)$ $-\frac{1}{10}\cot^5(2x)+C.$

13. Find $\int_{-1}^{1}(\sin^{11}x)dx.$ **Solution:** Here we do not need to use the technique of section 7.2. Since the interval is symmetric and the integrand is an odd function (check), then $\int_{-1}^{1}(\sin^{11}x)dx = 0.$

14. Find $\int \sqrt{1+9x^2}\,dx.$ **Solution:** Let I denote the given integral. First let $y = 3x.$ Then $dy = 3dx.$ We get $I = \frac{1}{3}\int \sqrt{1+y^2}\,dy.$ Let $y = \tan\theta, \frac{-\pi}{2} < \theta < \frac{\pi}{2}.$ Then $dy = (\sec^2\theta)d\theta$ and $\sqrt{1+y^2} = \sqrt{1+\tan^2\theta} = \left|\sec\theta\right| = \sec\theta$ since θ lies in the first or fourth quadrant. We get $I = \frac{1}{3}\int (\sec^3\theta)d\theta.$ Now we use integration by parts: Let $u = \sec\theta,$ $dv = (\sec^2\theta)d\theta.$ Then $du = (\sec\theta\tan\theta)d\theta,$ $v = \tan\theta.$ This gives $I = \frac{1}{3}[\sec\theta\tan\theta - \int (\sec\theta\tan^2\theta)d\theta] = \frac{1}{3}[\sec\theta\tan\theta - \int \sec^3(\theta)d\theta + \int (\sec\theta)d\theta]$ $= \frac{1}{3}[\sec\theta\tan\theta - 3I + \ln|\sec\theta+\tan\theta|].$ Solving for $I,$ we get $I = \frac{1}{6}[\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|]+C.$ In terms of $x,$ we get $I = \frac{1}{6}[3x\sqrt{1+9x^2} + \ln|\sqrt{1+9x^2}+3x|]+C.$

15. Find $\int \frac{1}{e^x - 2e^{-x}}dx.$ **Solution:** Multiply the integrand by $\frac{e^x}{e^x}$ to get $\int \frac{e^x}{e^{2x}-2}dx.$ Let $u = e^x.$ Then $du = e^x dx.$ The last integral becomes $\int \frac{1}{u^2-2}du = \frac{1}{2\sqrt{2}}\ln\left|\frac{u-\sqrt{2}}{u+\sqrt{2}}\right| + C = \frac{1}{2\sqrt{2}}\ln\left|\frac{e^x-\sqrt{2}}{e^x+\sqrt{2}}\right| + C.$

16. Find the limit of the sequence $\{\frac{\cos(5n)}{2+\sqrt[3]{n}}\}_{n=1}^{+\infty}.$ **Solution:** We use the Squeezing Theorem. Since $-1 \le \cos(5n) \le 1,$ then $\frac{-1}{2+\sqrt[3]{n}} \le \frac{\cos(5n)}{2+\sqrt[3]{n}} \le \frac{1}{2+\sqrt[3]{n}}.$ Since $\lim_{n\to+\infty}\frac{-1}{2+\sqrt[3]{n}} = 0$ and $\lim_{n\to+\infty}\frac{1}{2+\sqrt[3]{n}} = 0,$ then $\lim_{n\to+\infty}\frac{\cos(5n)}{2+\sqrt[3]{n}} = 0.$

17. Determine whether the series $\sum_{n=1}^{+\infty}\frac{1}{4n^2-1}$ converges or diverges. If it converges , find its sum. **Solution:** By definition, we need to find $\lim_{n\to+\infty} S_n$. We have $S_n=\sum_{k=1}^{n}\frac{1}{4k^2-1}=\frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{2k-1}-\frac{1}{2k+1}\right)=\frac{1}{2}[(\frac{1}{1}-\frac{1}{3})+(\frac{1}{3}-\frac{1}{5})+(\frac{1}{5}-\frac{1}{7})+...+(\frac{1}{2n-1}-\frac{1}{2n+1})]=\frac{1}{2}(1-\frac{1}{2n+1})$. This implies that $\lim_{n\to+\infty} S_n=\frac{1}{2}$. We conclude that the series converges and its sum is $\frac{1}{2}$.

18. Find all values of x for which the series $\sum_{n=0}^{+\infty}e^{-xn}$ converges and find the sum of the series for these values of x. **Solution:** The series $\sum_{n=0}^{+\infty}(e^{-x})^n=1+e^{-x}+(e^{-x})^2+...$ is a geometric series with $a=1$ and $r=e^{-x}$. Thus, it converges if $|r|<1$, i.e., $|e^{-x}|<1\Rightarrow e^{-x}<1\Rightarrow e^{x}>1\Rightarrow x>0$. The series converges if and only if $x>0$ and the sum in this case is $\frac{a}{1-r}=\frac{1}{1-e^{-x}}=\frac{e^x}{e^x-1}$.

19. Determine whether the series $\sum_{n=1}^{+\infty}\left(1-\frac{1}{n}\right)^{2n}$ converges or diverges. **Solution:** We use the Divergence Test: since $\lim_{n\to+\infty}\left(1-\frac{1}{n}\right)^{2n}=e^{-2}\neq 0$, then the given series diverges.

20. Determine whether the series $\sum_{n=2}^{+\infty}\frac{1}{n\sqrt{\ln n}}$ converges or diverges. **Solution:** We use the Integral Test: $f(x)=\frac{1}{x\sqrt{\ln x}}$ is positive, continuous and decreasing $(f'(x)=\frac{-((\ln x)^{-1/2}+2\sqrt{\ln x})}{2(x\sqrt{\ln x})^2}<0)$ on $[2,+\infty)$. So we can apply the Integral Test:
$$\int_2^{+\infty}\frac{1}{x\sqrt{\ln x}}dx=\lim_{t\to+\infty}\int_2^{t}\frac{1}{x\sqrt{\ln x}}dx=\lim_{t\to+\infty}2\sqrt{\ln x}\,|_2^t=\lim_{t\to+\infty}2\sqrt{\ln t}-2\sqrt{\ln 2}=+\infty.$$
Since the improper integral diverges, then the given series diverges.

21. Determine whether the series $\sum_{n=1}^{+\infty}\frac{1}{n\sqrt{n}}$ converges or diverges. **Solution:** The series can be rewritten as $\sum_{n=1}^{+\infty}\frac{1}{n\sqrt{n}}=\sum_{n=1}^{+\infty}\frac{1}{n^{3/2}}$. Thus the series is a p-series with $p=\frac{3}{2}>1$ and hence it is convergent.

22. Determine whether the series $\sum_{n=1}^{+\infty}\frac{\tan^{-1}(1/n)}{n^3}$ converges or diverges. **Solution:** We will use the Comparison Test. Since $\tan^{-1}(1/n)\le\pi/2$ for all $n\ge1$, then $\frac{\tan^{-1}(1/n)}{n^3}\le\frac{\pi}{2n^3}$ for all $n\ge1$. Since $\sum_{n=1}^{+\infty}\frac{\pi}{2n^3}$ converges (as it is a constant times a p-series with $p=3>1$), then the series $\sum_{n=1}^{+\infty}\frac{\tan^{-1}(1/n)}{n^3}$ converges.

23. Determine whether the series $\sum_{n=1}^{+\infty}\frac{2+3^n}{4+5^n}$ converges or diverges. **Solution:** We use the Limit Comparison Test. Choose $b_n=\frac{3^n}{5^n}=\left(\frac{3}{5}\right)^n$. Then $\lim_{n\to+\infty}\frac{a_n}{b_n}=$

$$\lim\ \frac{2+3^n}{4+5^n}\cdot\frac{5^n}{3^n}=\lim\frac{2\cdot5^n+15^n}{4\cdot3^n+15^n}=\lim\frac{15^n(2\cdot\frac{1}{3^n}+1)}{15^n(4\cdot\frac{1}{5^n}+1)}=\lim\frac{2\cdot\frac{1}{3^n}+1}{4\cdot\frac{1}{5^n}+1}=\frac{0+1}{0+1}=1>0.$$

Since the series $\sum_{n=1}^{+\infty}b_n=\sum_{n=1}^{+\infty}\left(\frac{3}{5}\right)^n$ converges (a geometric series with $|r|=\frac{3}{5}<1$), then the series $\sum_{n=1}^{+\infty}\frac{2+3^n}{4+5^n}$ converges by the LCT.

24. Determine whether the series $\sum_{n=1}^{+\infty}\frac{\cos(n\pi)}{\sqrt{n}}$ converges or diverges. **Solution:** Note first that $\cos(n\pi)=(-1)^n$. Thus, the series $\sum_{n=1}^{+\infty}\frac{\cos(n\pi)}{\sqrt{n}}=\sum_{n=1}^{+\infty}\frac{(-1)^n}{\sqrt{n}}$ is an alternating series. So we use the AST: $a_n=\frac{1}{\sqrt{n}}$; (i) $\lim_{n\to+\infty}\frac{1}{\sqrt{n}}=0$, and (ii) $f(x)=\frac{1}{\sqrt{x}}\Rightarrow f'(x)=\frac{-1}{2\sqrt{x^3}}<0$ for $x\ge1$, and hence $\left\{\frac{1}{\sqrt{n}}\right\}$ is decreasing. We conclude that the series $\sum_{n=1}^{+\infty}\frac{\cos(n\pi)}{\sqrt{n}}$ converges by the AST.

25. Determine whether the series $\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n^{0.3}}$ is absolutely convergent, conditionally convergent, or divergent. **Solution:** We check each possibility:

   a. <u>Absolute Convergence:</u> Since the series $\sum_{n=1}^{+\infty}\left|\frac{(-1)^{n+1}}{n^{0.3}}\right|=\sum_{n=1}^{+\infty}\frac{1}{n^{0.3}}$ diverges (a p-series with $p=0.3<1$), then the given series is not absolutely convergent.

   b. <u>Conditional Convergence:</u> We have to check the two conditions of conditional convergence:

i. $\sum_{n=1}^{+\infty}\left|\frac{(-1)^{n+1}}{n^{0.3}}\right|$ diverges: This was shown in (a).

ii. $\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n^{0.3}}$ converges: It is an alternating series and so we use the AST. We leave it to the reader to check that the series converges. **Conclusion:** The given series converges conditionally.

26. Determine whether the series $\sum_{n=1}^{+\infty}\frac{(-3)^{n+1}}{(3n+2)!}$ converges or diverges. **Solution:** We use the Ratio Test: $a_n = \frac{(-3)^{n+1}}{(3n+2)!}$, $a_{n+1} = \frac{(-3)^{n+2}}{(3(n+1)+2)!} = \frac{(-3)^{n+2}}{(3n+5)!}$.

$$L = \lim_{n\to+\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim\left|\frac{(-3)^{n+2}}{(3n+5)!}\cdot\frac{(3n+2)!}{(-3)^{n+1}}\right| = \lim\left|\frac{(-3)\cdot(3n+2)!}{(3n+5)(3n+4)(3n+3)\cdot(3n+2)!}\right|$$

$= \lim_{n\to+\infty}\frac{3}{(3n+5)(3n+4)(3n+3)} = 0.$ Since $L<1,$ then the series converges absolutely and hence the series converges.

27. Determine whether the series $\sum_{n=1}^{+\infty}\left(\frac{n+3}{n}\right)^{n^2}$ converges or diverges. **Solution:** We use the Root Test: $L = \lim\sqrt[n]{|a_n|} = \lim\sqrt[n]{\left(\frac{n+3}{n}\right)^{n^2}} = \lim_{n\to+\infty}\left(1+\frac{3}{n}\right)^n =$ e^3. Since $L>1,$ then the series diverges.

28. Find the sum of the series $\sum_{n=0}^{+\infty}\frac{1}{2^n\cdot n!}$. **Solution:** We compare the given series with the Maclaurin series of e^x: $e^x = \sum_{n=0}^{+\infty}\frac{x^n}{n!}$, $|x|<+\infty.$ If we substitute $x=\frac{1}{2}$ in the Maclaurin series of e^x, we get $\sum_{n=0}^{+\infty}\frac{1}{2^n\cdot n!} = e^{1/2} = \sqrt{e}.$