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# Math 101- Solved Problems
## (16 problems, 4 pages)

**Q1.** Find the limit $\lim_{x\to -1^-} \dfrac{|x+1|}{x^2-1}$. [Final Exam, Term 072]

**Solution:** As $x \to -1^-$, $x < -1$ and so $|x+1| = -(x+1)$. This gives

$$\lim_{x\to -1^-} \frac{|x+1|}{x^2-1} = \lim_{x\to -1^-} \frac{-(x+1)}{(x-1)(x+1)} = \lim_{x\to -1^-} \frac{-1}{x-1} = \frac{1}{2}.$$

**Q2.** Find the limit $\lim_{x\to 1} \dfrac{x^3-1}{\sqrt{2x+2}-2}$. [Exam I, Term 083]

**Solution:** First multiply by the conjugate, then factor and simplify.

$$\lim_{x\to 1} \frac{x^3-1}{\sqrt{2x+2}-2}\cdot\frac{\sqrt{2x+2}+2}{\sqrt{2x+2}+2} = \lim_{x\to 1} \frac{(x^3-1)(\sqrt{2x+2}+2)}{(2x+2)-4}$$

$$= \lim_{x\to 1} \frac{(x-1)(x^2+x+1)(\sqrt{2x+2}+2)}{2(x-1)}$$

$$= \lim_{x\to 1} \frac{1}{2}(x^2+x+1)(\sqrt{2x+2}+2)$$

$$= \frac{1}{2}(3)(4) = 6.$$

**Q3.** Find the limit $\lim_{x\to +\infty} \dfrac{\cos^2 x}{x^3}$. [Exam I, Term 072]

**Solution:** We use the Squeeze Theorem. First note that

$$0 \le \cos^2 x \le 1.$$

As $x \to +\infty$, $x^3 > 0$. By dividing the above inequalities by $x^3$, we get

$$0 \le \frac{\cos^2 x}{x^3} \le \frac{1}{x^3}.$$

As $\lim_{x\to +\infty} 0 = 0 = \lim_{x\to +\infty} \dfrac{1}{x^3}$, then, by the Squeeze Theorem, $\lim_{x\to +\infty} \dfrac{\cos^2 x}{x^3} = 0$.

**Q4.** Find the limit $\lim_{x\to 6^+} \tan^{-1}(\ln(x-6))$. [Exam I, Term 082]

**Solution:** As $x \to 6^+$, $\ln(x-6) \to -\infty$ (check the graph of $y = \ln(x-6)$, and hence $\tan^{-1}(\ln(x-6)) \to -\pi/2$.

**Q5.** Let $f(x) = \dfrac{ax+b}{cx+d}$, where $a,b,c,d$ are constants. Find $f'(x)$. [Exam II, Term 073]

**Solution:** By the quotient rule, we get

$$f'(x) = \frac{(cx+d)\cdot a - (ax+b)\cdot c}{(cx+d)^2} = \frac{ad-bc}{(cx+d)^2}.$$

**Q6.** Find the limit $\lim_{t\to 0}\dfrac{\tan t-\sin t\cos t}{t\sin^2 t}$. [Exam II, Term 072]

**Solution:** Write $\tan t=\dfrac{\sin t}{\cos t}$ and simplify:

$$\lim_{t\to 0}\frac{\frac{\sin t}{\cos t}-\sin t\cos t}{t\sin^2 t}=\lim_{t\to 0}\frac{\sin t\cdot(1-\cos^2 t)}{t\cos t\sin^2 t}=\lim_{t\to 0}\frac{\sin t}{t}\cdot\frac{1}{\cos t}=1\cdot 1=1$$

**Q7.** Find the limit $\lim_{\theta\to 0}\dfrac{\cos\theta-1}{3\tan^2\theta}$. [Exam II, Term 081]

**Solution:** First multiply by the conjugate and then simplify:

$$\lim_{\theta\to 0}\frac{\cos\theta-1}{3\tan^2\theta}\cdot\frac{\cos\theta+1}{\cos\theta+1}=\lim_{\theta\to 0}\frac{-\sin^2\theta}{3\tan^2\theta\cdot(\cos\theta+1)}$$

$$=\lim_{\theta\to 0}\frac{-\cos^2\theta}{3(\cos\theta+1)}=\frac{-1}{6}.$$

**Q8.** Find the slope of the tangent line to the curve $y=(1-x^{-1})^{-1}$ at $x=-1$. [Exam II, Term 072]

**Solution:** It is equal to $\dfrac{dy}{dx}\Big|_{x=-1}$. By the chain rule, $\dfrac{dy}{dx}=(-1)(1-x^{-1})^{-2}(0+x^{-2})$.

Thus $\dfrac{dy}{dx}\Big|_{x=-1}=(-1)(2)^{-2}(1)=\dfrac{-1}{4}$.

**Q9.** If $L(x)=(f\circ g\circ h)(x)$, where $h(1)=2, g(2)=3, h'(1)=4, g'(2)=5,$ and $f'(3)=6,$ then find $L'(1)$. [Exam II, Term 083]

**Solution:** As $L(x)=f(g(h(x))),$ then using the chain rule, we have

$$L'(x)=f'(g(h(x))\cdot g'(h(x))\cdot h'(x)$$

Thus we get

$$\begin{aligned}L'(1)&=f'(g(h(1)))\cdot g'(h(1))\cdot h'(1)\\&=f'(g(2))\cdot g'(2)\cdot 4\\&=f'(3)\cdot 5\cdot 4\\&=6\cdot 5\cdot 4=120\end{aligned}$$

**Q10.** Find $y'$ if $y\sin x=x^3+\tan y$. [Exam II, Term 082]

**Solution:** By implicit differentiation, we get

$$y\cos x+y'\sin x=3x^2+\sec^2 y\cdot y'$$

Solving for $y'$, we obtain

$$y'=\frac{y\cos x-3x^2}{\sec^2 y-\sin x}.$$

**Q11.** Find $\dfrac{dy}{dx}$ if $y=\ln\sqrt{\dfrac{1+\sin x}{1-\sin x}}$. [Exam II, Term 073]

**Solution:** First simplify the function:

$$y = \frac{1}{2}\left(\ln(1+\sin x) - \ln(1-\sin x)\right).$$

Differentiating and then simplifying, we get

$$\frac{dy}{dx} = \frac{1}{2}\left[\frac{\cos x}{1+\sin x} - \frac{-\cos x}{1-\sin x}\right] = \frac{1}{2}\left[\frac{\cos x\,(1-\sin x) + \cos x\,(1+\sin x)}{1-\sin^2 x}\right]$$

$$= \frac{1}{2}\left[\frac{2\cos x}{\cos^2 x}\right] = \frac{1}{\cos x} = \sec x.$$

**Q12.** Find the value of $\cosh\left(\frac{1}{3}\ln x\right) + \sinh\left(\frac{1}{3}\ln x\right)$. [Exam II, Term 083]

**Solution:** Since $\cosh z + \sinh z = e^z$, then

$$\cosh\left(\frac{1}{3}\ln x\right) + \sinh\left(\frac{1}{3}\ln x\right) = e^{\left(\frac{1}{3}\ln x\right)} = e^{\ln \sqrt[3]{x}} = \sqrt[3]{x}.$$

**Q13.** Find $\frac{dy}{dx}$ if $y = x \sinh^{-1}\left(\frac{x}{3}\right) - \sqrt{9+x^2}$. [Exam II, Term 073]

**Solution:** Use the product rule for the first term.

$$\frac{dy}{dx} = x \cdot \frac{1}{\sqrt{1+(x/3)^2}} \cdot \frac{1}{3} + \sinh^{-1}\left(\frac{x}{3}\right) \cdot 1 - \frac{1}{2\sqrt{9+x^2}} \cdot 2x$$

$$= \frac{x}{\sqrt{9+x^2}} + \sinh^{-1}\left(\frac{x}{3}\right) - \frac{x}{\sqrt{9+x^2}} = \sinh^{-1}\left(\frac{x}{3}\right).$$

**Q14.** Find the linear approximation of $f(x) = e^{-x^2}$ at $a = 0$. [Final Exam, Term 072]

**Solution:** It is given by

$$f(x) \approx f(0) + f'(0)(x-0).$$

Now $f(0) = 1$. Since $f'(x) = e^{-x^2}(-2x)$, then $f'(0) = 0$. Thus the linear approximation of $f(x) = e^{-x^2}$ at $a = 0$ is given by

$$e^{-x^2} \approx 1, \quad \text{when } x \text{ is close to } 0.$$

**Q15.** Find the critical numbers of $f(x) = \sqrt[3]{x^2 - x}$. [Final Exam, Term 082]

**Solution:** We first find $f'$: $f'(x) = \frac{1}{3}(x^2 - x)^{\frac{-2}{3}}(2x-1) = \frac{1}{2} \cdot \frac{2x-1}{(x^2-x)^{2/3}}$.

- $f'(x) = 0 \Rightarrow (2x-1) = 0 \Rightarrow x = \frac{1}{2}$
- $f'(x)$ DNE when $(x^2 - x)^{2/3} = 0$; i.e., when $x^2 - x = 0$. This implies that $x = 0$ or $x = 1$.

As all of the values $x = 0, \frac{1}{2}, 1$ are in the domain of $f$, then the critical numbers of $f$ are $0$, $\frac{1}{2}$, and $1$.

**Q16.** If $M$ and $m$ are respectively the absolute maximum and absolute minimum values of $f(x) = x\sqrt{4-x^2}$ on $[-1, 2]$, then find the value of $\sqrt{3}M + 4m$. [Final Exam, Term 081]

**Solution:** Since $f$ is continuous on $[-1, 2]$, then it has absolute extrema on the given interval. We find the critical numbers of $f$:

$$f'(x) = x\cdot\frac{1}{2}(4-x^2)^{-1/2}\cdot(-2x) + \sqrt{4-x^2}\cdot 1 = (4-x^2)^{-1/2}[-x^2 + (4-x^2)]$$

$$= \frac{4-2x^2}{\sqrt{4-x^2}}$$

We have the following:

- $f'(x) = 0 \Rightarrow 4-2x^2 = 0 \Rightarrow x = \pm\sqrt{2}$. But $x = -\sqrt{2} \notin (-1, 2)$. So we take $x = \sqrt{2}$ only.
- $f'(x)$ DNE when $\sqrt{4-x^2} = 0$; that is, when $x = \pm 2$. Since $x = \pm 2 \notin (-1, 2)$, then the values are rejected.

Now we calculate the value of $f$ at $x = \sqrt{2}, -1, 2$:

$$f(\sqrt{2}) = 2,\ f(-1) = -\sqrt{3},\ f(2) = 0.$$

Thus we get $M = 2$ and $m = -\sqrt{3}$ and hence $\sqrt{3}M + 4m = -2\sqrt{3}$.