Magic Squares

Introduction

Numbers have always fascinated people. The strange and surprising properties numbers have either individually or in relation with each other force us to believe that numbers have intrinsic beauty and magic. It is so amazing how ordering numbers in certain ways leads to very surprising objects. In this article, we present one such a magical object; namely, what is known as **a magic square**.

What is a magic square

A magic square of order N is a square consisting of N rows and N columns that is filled with distinct integers in such a way that the sum of the integers in each row, each column and each main diagonal is equal to some fixed number. This fixed number is called **the magic sum** of the square. Each space of the square is called **a cell**. A magic square of order N consists of N^2 cells. Figure 1 is an example of a magic square of order 3 that has 9 cells and has a magic sum of 15.

8	1	6
3	5	7
4	9	2

Magic sum = 15
Figure 1

11	4	9
6	8	10
7	12	5

Magic sum = 24
Figure 2

71	89	17
5	59	113
101	29	47

Magic sum = 177
Figure 3

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Magic sum = 34
Figure 4

In a magic square of order N, the integers used are usually the consecutive integers $1,2,3,...,N^2$. In this case, the square is called a **pure magic square** or a **traditional magic square**. Figure 1 and Figure 4 are examples of pure magic squares. Figure 2 and Figure 3 are examples of magic squares that are not pure.

How to calculate the magic sum

We will denote the magic sum of a magic square by the letter S. The magic sum of any magic square is given by the following relation

$$S = (\text{sum of the numbers used}) \div (\text{the order of the square})$$

For a pure magic square of order N, the sum of the numbers used is

$$1+2+3+...+N^2 = \frac{1}{2}N^2(N^2+1).$$

If we divide the above quantity by N, the order of the square, we get the following formula for the sum of a pure magic square of order N:

$$S = \frac{1}{2} N(N^2 + 1).$$

For example, if $N = 3$, then the magic sum is $S = \frac{1}{2} \times 3(3^2 + 1) = 15$. Similarly, if $N = 4$, then $S = 34$; if $N = 5$, then $S = 65$.

Constructing Pure Magic Squares

There are different methods for constructing magic squares. The methods used depend on the order of the square. In the literature, magic squares are divided according to their orders into three classes:

1. Magic squares of odd orders. These are the ones whose order is not divisible by 2. Mathematically, this order can be written in the form $N = 2m + 1$, where m is some integer. Examples of odd orders are 3, 5 and 7.
2. Magic squares of singl-even orders. These are the ones whose order is even and is divisible by 2 and not by 4. The general formula for this order is $N = 2(2m + 1)$, where m is some integer. Examples of this order are 6, 15 and 14.
3. Magic squares of doubl-even orders. These are the ones whose order is even and is divisible by 2 and 4.The general formula for such an order is $N = 4m$, where m is some integer. Examples of this order are 4, 8 and 12.

In what follows we will describe three different methods, one method for each one of the above three classes, for constructing pure magic squares.

Constructing Pure Magic Squares of Odd Orders

There are several methods for constructing pure magic squares of odd orders. In this section, we describe one simple method. This method consists of the following two steps:

1. Write 1 in the center cell of the first (top) row. The numbers $2, 3, ..., N^2$ are then inserted in this increasing order in the remaining cells of the square according to the rules given in step 2 below.
2. Move from one cell to another cell by going up and then to the left, one step at a time. In moving this way, we have the following possibilities:
    a. If a move takes you to another empty cell in the square, then insert the number in that cell.
    b. If a move takes you out at the top of a column of the square, then insert the number in the bottom cell of that column.
    c. If a move takes you out at the left of a row of the square, then insert the number in the last cell to the right of that row.
    d. If a move takes you to a cell that is already filled, then insert the number in the cell that is immediately below the cell you started with.

e. If a move takes you to the upper left corner of the square, then insert the number in this cell **and** then insert the next number in the cell that is immediately below the upper left corner cell.

The following figure illustrates how a pure magic square of order 3 is filled by following the above steps.

	1	

	1	
2		

	1	
		3
2		

	1	
		3
2		4

	1	
	5	3
2		4

6	1	
	5	3
2		4

6	1	
7	5	3
2		4

6	1	8
7	5	3
2		4

6	1	8
7	5	3
2	9	4

Figure 5

Following the above steps, we can also construct a pure magic square of order 5. See Figure 6.

15	8	1	24	17
16	14	7	5	23
22	20	13	6	4
3	21	19	12	10
9	2	25	18	11

Magic sum = 65

Figure 6

Constructing Pure Magic Squares of Single-even Order

This type of squares is considered the hardest to construct. One method to construct pure magic squares of single-even order is to use the method invented by Phillipe de la Hire (1640-1719). **De la Hire's method** is described as follows:

1. We construct two squares and then add them cell-wise to get the required magic square.
2. The first square is constructed as follows:
   a. Fill the main diagonals with the numbers $1,2,3,...,N$, in this order, starting from the upper and lower left corners. See Figure (a) for the case of $N = 6$.
   b. In general, two cells of column i are filled with i. For each $i = 1,2,...,N/2$, do the following:
      i. Complete filling column i using i and its complement $N - i + 1$ in such a way that the number of cells containing i is the same as the number of cells containing $N - i + 1$.

ii. Fill the unoccupied cells of column $N-i+1$ by the complements of the numbers in the opposite cells of column i. This gives the first square. Make sure that this square does not contain two identical rows. See Figure (b).

3. The second square is constructed as follows:
   a. Construct a new square in such a way that the elements of row i are the elements of column i of the first square already constructed in step 2. See Figure (c).
   b. Replace each number of the previous square by its deficient multiple; the deficient multiple of i is $(i-1)N$. This gives the second square. See Figure (d).
4. Add the two resulting squares (cell-wise) to get the required magic square. See Figure 7 for a pure magic square of order 6 whose magic sum is 111.

1					6
	2			5	
		3	4		
		3	4		
	2			5	
1					6

(a)

1	5	4	3	2	6
6	2	3	4	5	1
6	5	3	4	2	1
1	5	3	4	2	6
6	2	4	3	5	1
1	2	4	3	5	6

(b)

1	6	6	1	6	1
5	2	5	5	2	2
4	3	3	3	4	4
3	4	4	4	3	3
2	5	2	2	5	5
6	1	1	6	1	6

(c)

0	30	30	0	30	0
24	6	24	24	6	6
18	12	12	12	18	18
12	18	18	18	12	12
6	24	6	6	24	24
30	0	0	30	0	30

(d)

1	35	34	3	32	6
30	8	27	28	11	7
24	17	15	16	20	19
13	23	21	22	14	18
12	26	10	9	29	25
31	2	4	33	5	36

Figure 7**:** (b) + (d)

For the case of $N=6$ shown above, observe that the sum of the numbers in each row, each column, and each main diagonal in Figure (b) is equal to 21 and in Figure (d) is equal to 90. When we add cell-wise the squares in Figures (b) and (d), we get a pure magic square whose magic sum is 111=21+90. In the general case, the corresponding sums in Figures (b) and (d) are $\frac{1}{2}N(N+1)$ and $\frac{1}{2}N^2(N-1)$, respectively. Adding these two sums gives $\frac{1}{2}N(N^2+1)$, which is the magic sum of a pure magic square of order N.

Constructing Pure Magic Squares of Double-even Order

For this type of squares, we will use Durer's method. This method was invented by Albrecht Durer in 1514. **Durer's method** is described as follows:

1. Draw five empty squares of the required order.
2. In the first square, fill in the cells of the square by the numbers $1,2,...,N^2$ by going horizontally to the right in each row (starting with the 1st row, then the 2nd row, etc.)
3. In the second square, fill in the cells of the main diagonals by the numbers in the cells of the main diagonals of the first square.
4. In the third square, fill in the cells of the square by the numbers $1,2,...,N^2$ by going horizontally to the left in each row (starting with the last row, then the N-1st row, etc.)
5. In the fourth square, fill in the cells not lying on the main diagonals by exactly the numbers not lying on the main diagonals of the third square.
6. Combine the second and fourth squares in the fifth square to get the required pure magic square.

Figure 8 illustrates the above steps for constructing a pure magic square of order 4.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

1			4
	6	7	
	10	11	
13			16

16	15	14	13
12	11	10	9
8	7	6	5
4	3	2	1

	15	14	
12			9
8			5
	3	2	

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

Figure 8

Once the method is understood, one can do the above five steps in one shot. We ask the reader to compare Figure 4 and Figure 8.

Pure Magic Squares of Order One and Two

Using the three methods described above, one can construct pure magic square of any order larger than 2. It remains to consider magic squares of order 1 and 2. There is only one possibility for a pure magic square of order 1; namely, the square consisting of one cell that is filled with 1. There is no pure magic square of order 2. In this case, the numbers used are 1, 2, 3, 4 and the magic sum is 5. In any way we want to fill in the square, there will always be a row, a column or a diagonal that contains only 1 and 2. These two numbers do not add up to 5.

New Magic Squares from Old Ones

Starting with a magic square, we can construct a new magic square by simply using the four basic arithmetic operations of addition, subtraction, multiplication and division. If we add a number to each cell of a given magic square, then we get a new magic square of the same order, but with a different magic sum. For example, if we add 4 to each cell of the magic square given in Figure 1, then we get the magic square shown in Figure 9. Similarly, subtracting a number from each cell of a given magic square will yield a new magic square. If we multiply or divide each cell of a magic square by some number, we also get new magic squares. In case of division, the resulting numbers in the magic square will be no longer integers. Also, the resulting magic squares are not pure. Figures 9-12 give some magic squares of order 3 originated from Figure 1.

12	5	10
7	9	11
8	13	6

Magic sum= 19
Figure 9

5	-2	3
0	2	4
1	6	-1

Magic sum = 12
Figure 10

24	3	18
9	15	21
12	27	6

Magic sum = 45
Figure 11

4	1/2	3
3/2	5/2	7/2
2	9/2	1

Magic sum = 15/2
Figure 12

Final Words

It was a short journey through magic squares. More methods of constructing magic squares and more magical properties of magic squares can be found in the references given below.

References

1. Andrews, W. S., Magic Squares and Cubes, Cosimo Classics, New York, 2004.
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3. Pickover, C. A., The Zen of Magic Squares, Circles, and Stars, Princeton University Press, Princeton, 2002.

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