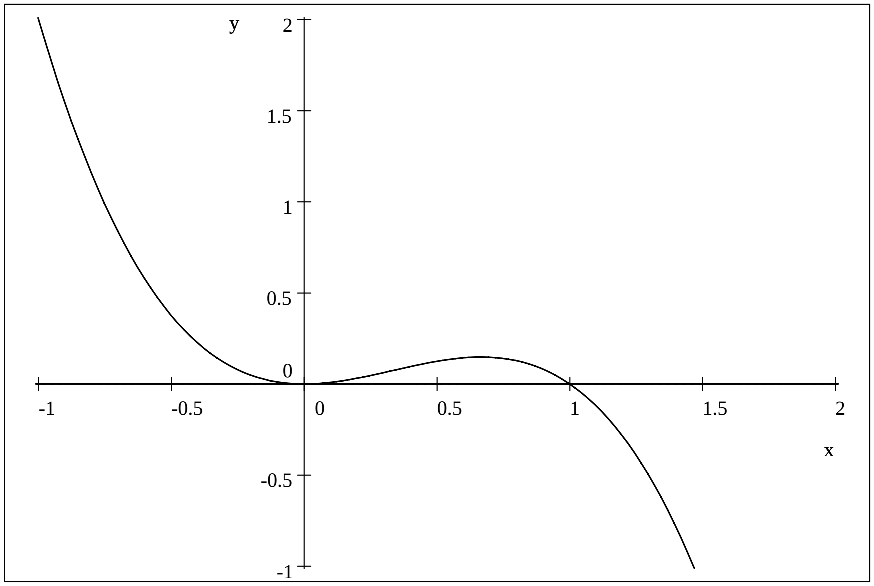
Exam II. Math 102 Term (082) .Monday, May 18, 2009. Marks: 100. Time: 2 Hours.

Q.1. $(7\ points)$. Find the volume of the solid obtained by rotating the region bounded by $y = x^2 - x^3$ and $y = 0$ about the $y - axis$.

SOLUTION. *. Intersection with $X - axis$ and $Y-$ axis.

$y = x^2 - x^3$



$y(0) = 0$

$y(x) = x^2 - x^3 = 0$

$\Longrightarrow x^2(1-x) = 0$

$\Longrightarrow x = 0$ or $x = 1$..... (2 points)

**. Rotate the region R about $Y - axis$,

the volume of the solid obtained is given by:

$V = 2\pi \int_0^1 x\left(x^2 - x^3\right) dx$...... (3 points)

$= 2\pi \int_0^1 \left(x^3 - x^4\right) dx$

$= 2\pi \left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1$...... (1 point)

$= 2\pi \left[x^4\left(\frac{1}{4} - \frac{x}{5}\right)\right]_0^1$

$= 2\pi \left[(1)^4\left(\frac{1}{4} - \frac{1}{5}\right)\right]$

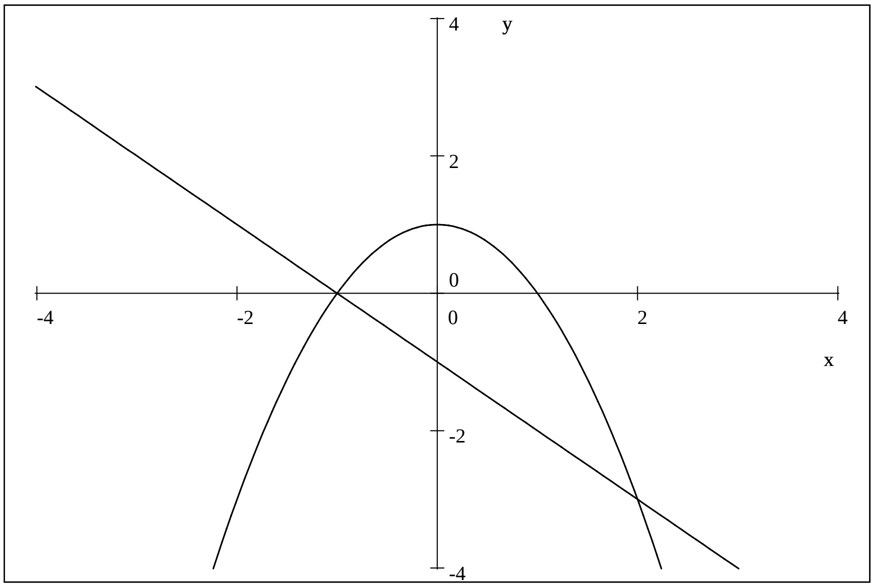
$= 2\pi \left[1\left(\frac{1}{20}\right)\right] = \frac{\pi}{10}$...... (1 point)

Q.2. (9 *points*). Find the volume of the solid obtained by rotating the region bounded by the graphs of

$y = 1 - x^2$ and $y = -(1 + x)$

about the line $x = 3$.

Solution. $y = 1 - x^2$



I. Points of intersection

$1 - x^2 = -(1 + x)$

$\Longrightarrow x^2 - x - 2 = 0$

$\Longrightarrow (x + 1)(x - 2) = 0$

$\Longrightarrow x = -1,\ x = 2$......... (2 points)

II. Vol me of Shell $= 2\pi(3 - x_i)\left[(1 - x_i^2) + (1 + x_i)\right] \triangle x.$

$III.$ Volume of Solid $= 2\pi \int_{-1}^{2} (3 - x)\left[(1 - x^2) + (1 + x)\right] dx$..... (4 points)

IV. $F(x) = (3 - x)(2 + x - x^2) = 6 + x - 4x^2 + x^3$

$V.\ Evaluate$ Integral.

$$\int_{-1}^{2} F(x)\,dx = \left[6x + \frac{x^2}{2} - \frac{4x^3}{3} + \frac{x^4}{4}\right]_{-1}^{2}$$

$$= \left(12 + 2 - \frac{32}{3} + 4\right) - \left(-6 + \frac{1}{2} + \frac{4}{3} + \frac{1}{4}\right)$$

$$= 24 - \frac{32}{3} - \frac{4}{3} - \frac{3}{4}$$

$$= 24 - 12 - \frac{3}{4} = \frac{45}{4}.$$

$VI.$ Volume of Solid $= 2\pi\left(\dfrac{45}{4}\right) = \dfrac{45\pi}{2}$..... (3 points)

Q.3. (4 points). Find the average value of the function $f(t) = t\sin(t^2)$ on the interval $[0, \sqrt{\pi}]$.

SOLUTION. $f_{aver} = \dfrac{1}{\sqrt{\pi}-0}\int_0^{\sqrt{\pi}} t\sin(t^2)\,dt$..... (2 points)

$= \dfrac{1}{\sqrt{\pi}}\int_0^{\pi} \dfrac{1}{2}\sin u du$

$$\left[\begin{array}{c} u = t^2, \\ du = 2tdt, \\ \frac{1}{2}du = tdt \end{array}\right]$$

$$\left[\begin{array}{c} t = 0, u = 0; \\ t = \sqrt{\pi}, u = \pi \end{array}\right]$$

$f_{aver} = \left[\dfrac{-1}{\sqrt{\pi}}.\dfrac{1}{2}(\cos u)\right]_0^{\pi}$..............(1 point)

$= \dfrac{1}{2\sqrt{\pi}}(-\cos\pi + \cos 0)$

$= \dfrac{1}{2\sqrt{\pi}}(1+1)$

$= \dfrac{2}{2\sqrt{\pi}}$

$= \dfrac{1}{\sqrt{\pi}}$........(1 point)

Q.4.Evaluate the integral or show that it is divergent.

(a) (5 points) . $\int_0^1 \ln x \; dx$

Solution. $\int_0^1 \ln x \; dx$

Using integration by Parts:

$\int \ln x \; dx$

$= \int (\ln x.1) \; dx$

$= \ln x \int 1dx - \int \left[\frac{d}{dx}(\ln x) . \int 1dx\right] dx$

$= (\ln x)\, x - \int \left(\frac{1}{x}\right) .xdx$

$= x \ln x - \int dx$

$= x \ln x - x + C$........(2 points)

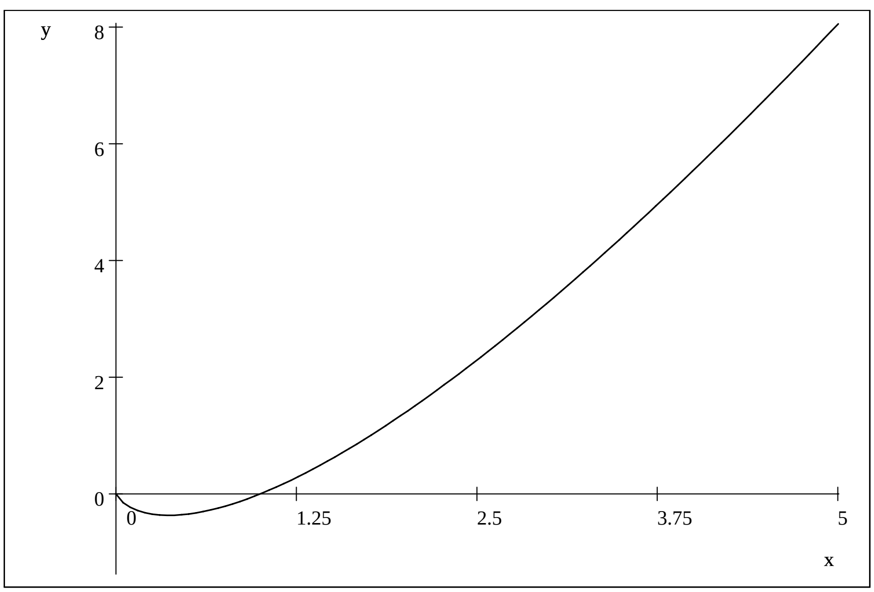
Now $\int_0^1 \ln x \; dx$

$= [x \ln x - x]_{x=1} - \lim_{x \longrightarrow 0^+} [x \ln x - x]$

$= 1. \ln(1) - 1 - 0 = -1$.........(1 point)

$\lim_{x \longrightarrow 0^+} x \ln x = 0$.(L'sHopital Rule)........(2 points)

$y = x \ln x$.Note: Graph is not required.



Q.4. (b) . (4 points) . $\int_2^{+\infty} \dfrac{1}{x \ln x} dx$

Consider $I = \int \dfrac{1}{x \ln x} dx$

Let $u = \ln x$. Then $du = \dfrac{1}{x} dx$

Thus $I = \int \dfrac{1}{u} du$

$= \ln |u| + C$

$= \ln |\ln x| + C$........(2 points)

Thus $\int_2^{+\infty} \dfrac{1}{x \ln x} dx$

$= \lim\limits_{x \longrightarrow +\infty} [\ln |\ln x|] - [\ln |\ln x|]_{x=2}$........(1 point)

$= +\infty - \ln |\ln 2|$........(1 point)

$= +\infty$.

The integral is Divergent.

Q.5. Evaluate the following integrals:

(a) (5 points) . $\int x\tan^{-1}x\ dx$

Solution.

$Let\ u = \tan^{-1}x,$

then $du = \dfrac{dx}{1+x^2}$

$dv = xdx,\ v = \dfrac{x^2}{2}$

...................................(2 points)

Hence $\int x\tan^{-1}x\ dx$

$= \int udv = uv - \int vdx$

$= \dfrac{x^2}{2}\tan^{-1}x - \int \dfrac{x^2}{2}.\dfrac{1}{1+x^2}dx$.....(1 point)

$= \dfrac{1}{2}x^2\tan^{-1}x - \dfrac{1}{2}\int \dfrac{x^2}{1+x^2}dx$

$= \dfrac{1}{2}x^2\tan^{-1}x - \dfrac{1}{2}\left[\int\left(1 - \dfrac{1}{1+x^2}\right)dx\right]$

$= \dfrac{1}{2}x^2\tan^{-1}x - \dfrac{1}{2}x + \dfrac{1}{2}\tan^{-1}x + C$......(2 points)

Q.5.(b) (5 *points*) .$\int \tan^2 x \ \sec^4 x \ dx$.

SOLUTION.

Let $I = \int \tan^2 x \sec^4 x \ dx$.

Then $I = \int \tan^2 x \ \sec^2 x \ \sec^2 x \ dx$.....(1 point)

$= \int \tan^2 x \, (1 + \tan^2 x) \sec^2 x \ dx$.....(1 point)

Let $u = \tan x$. Then $du = \sec^2 x \ dx$.....(1 point)

$\Longrightarrow I = \int u^2 \, (1 + u^2) \, du$

$= \int (u^2 + u^4) \, du$

$= \frac{1}{3}u^3 + \frac{1}{5}u^5 + C$......(1 point)

Thus $I = \frac{1}{3} \tan^3 x + \frac{1}{5} \tan^5 \ x + C$......(1 point)

Q.5.(c). (8 points) . $\int \frac{(x-3)\,dx}{\sqrt{8x-x^2}}$.

SOLUTION. $*$. *Complete* square:

$8x - x^2 = -\left(x^2 - 8x\right)$

$= -\left(x^2 - 8x + 16\right) + 16$

$= 16 - (x-4)^2$.....(1 point)

$**$. $I = \int \frac{(x-3)\,dx}{\sqrt{8x-x^2}}$

$= \int \frac{(x-3)\,dx}{\sqrt{16-(x-4)^2}}$

$= \int \frac{(x-4)\,dx}{\sqrt{16-(x-4)^2}} + \int \frac{dx}{\sqrt{16-(x-4)^2}}$

Put $x - 4 = 4\sin\theta \Longrightarrow dx = 4\cos\theta d\theta$.....(2 points)

Then $I = \int \frac{4\sin\theta}{4\cos\theta}.4\cos\theta d\theta + \int \frac{4\cos\theta}{4\cos\theta} d\theta$

$= 4\int \sin\theta d\theta + \int d\theta = -4\cos\theta + \theta + C$.....(2 points)

Since $\cos^2\theta = 1 - \sin^2\theta$

So $\cos^2\theta = 1 - \left(\frac{x-4}{4}\right)^2 = 1 - \frac{(x-4)^2}{16}$

$= 1 - \frac{x^2-8x+16}{16} = \frac{8x-x^2}{16}$.

$\cos\theta = \frac{\sqrt{8x-x^2}}{4}$.....(2 points)

Thus $I = -\,4.\cos\theta + \sin^{-1}\left(\frac{x-4}{4}\right) + C$.....(1 point)

$= -\,4.\frac{\sqrt{8x-x^2}}{4} + \sin^{-1}\left(\frac{x-4}{4}\right) + C$

$= -\sqrt{8x-x^2} + \sin^{-1}\left(\frac{x-4}{4}\right) + C.$

Q.5.(d) . (8 points) . $\int \dfrac{dx}{3-5\sin x}$

Let $t=\tan\dfrac{x}{2}$. Then $dx=\dfrac{2dt}{1+t^2}$, $\sin x=\dfrac{2t}{1+t^2}$......(2 points)

So $\int \dfrac{dx}{3-5\sin x}=2\int \dfrac{dt}{(1+t^2)\,.\left(3-5.\dfrac{2t}{1+t^2}\right)}$

$=2\int \dfrac{dt}{3+3t^2-10t}$. Let $I=\int \dfrac{dt}{3+3t^2-10t}$......(1 point)

Then $I=\int \dfrac{dt}{(3t-1)(t-3)}$.......... (1 point)

$=\int \dfrac{-3/8}{(3t-1)}dt+\int \dfrac{dt}{t-3}$.....(2 points)

$=-\dfrac{1}{8}\int \dfrac{3}{3t-1}dt+\dfrac{1}{8}\int \dfrac{1}{t-3}dt$

$=-\dfrac{1}{8}\ln|3t-1|+\dfrac{1}{8}\ \ln|t-3|+C$.....(1 point)

$=-\dfrac{1}{8}\ln\left|3\tan\dfrac{x}{2}-1\right|+\dfrac{1}{8}\ln\left|\tan\dfrac{x}{2}-3\right|+C$.....(1 point)

$=\dfrac{1}{8}\ln\left|\dfrac{t-3}{3t-1}\right|$.

...................OR....................

$\int \dfrac{dx}{3-5\sin x}$

$=\dfrac{1}{4}\ln\left|\dfrac{\tan\left(\dfrac{x}{2}\right)-3}{3\tan\left(\dfrac{x}{2}\right)-1}\right|+C$......(1 point)

NOTE: Partial Fraction:

$$\frac{1}{(3t-1)(t-3)}=\frac{A}{3t-1}+\frac{B}{t-3}$$

$$=\frac{At-3A+3Bt-B}{(3t-1)(t-3)}=\frac{(A+3B)\,t+(-3A-B)}{(3t-1)(t-3)}$$

So $A+3B=0$, $-3A-B=1$

Solving: $A=-3/8$, $B=1/8$.

Q.5.(e) (9 points) . $\int \frac{x^2+7x+11}{(x^2+6x+13)(x-2)}dx.$

Let $\frac{x^2+7x+11}{(x^2+6x+13)(x-2)}$

$\frac{A}{x-2}+\frac{Bx+C}{x^2+6x+13}$.....(1 point)

$\Longrightarrow x^2+7x+11=$

$A(x^2+6x+13)+(Bx+C)(x-2)$.....(1 point)

$x=2 \Longrightarrow 29=29A \Longrightarrow A=1$.....(1 point)

$x=0 \Longrightarrow 11=13A-2C \Longrightarrow C=1$

$x=1 \Longrightarrow 19=20A+(B+C)(-1)$

$\Longrightarrow -1=(B+1)(-1) \Longrightarrow B=0$. (*up to here*).....(2 points)

So $\int \frac{x^2+7x+11}{(x^2+6x+13)(x-2)}dx$

$=\int \frac{1}{x-2}dx+\int \frac{1}{x^2+6x+13}dx$

$=\ln|x-2|+\int \frac{1}{(x+3)^2+4}dx$; .....(1 point)

$u=x+3.$

$=\ln|x-2|+\int \frac{1}{u^2+4}du$

$=\ln|x-2|+\frac{1}{4}\int \frac{1}{\left(\frac{u}{2}\right)^2+1}du$

$=\ln|x-2|+$.....................(1 point)

$\frac{1}{2}\tan^{-1}\left(\frac{x+3}{2}\right)+C$......(2 points)

Q.5. (f) (6 points) . $\int \frac{1}{x+\sqrt[3]{x}}dx.$

SOLUTION. *Let* $u = \sqrt[3]{x}$

$\Longrightarrow u^3 = x$, $3u^2du = dx$......(2 points)

So $\int \frac{1}{x+\sqrt[3]{x}}dx$

$= \int \frac{3u^2du}{u^3+u}$.....(1 point)

$= 3\int \frac{u^2}{u(u^2+1)}du$

$= 3\int \frac{u}{u^2+1}du$

$= \frac{3}{2}\int \frac{2u}{u^2+1}du$

$\left[\text{Si}\, nce\ \frac{d}{du}(u^2+1) = 2u\right]$

$= \frac{3}{2}\ln|u^2+1| + C$.....(2 points)

$= \frac{3}{2}\ \ln\left|\sqrt[3]{x^2}+1\right| + C$......(1 point) .

Q.6. Determine whether the sequence converges or diverges. If it converges, find its limit.

(a) . (4 points) . $a_n = \dfrac{1+\sin 2n}{1+\sqrt{n}}$.

Solution. By the squeeze Theorem, and since $0 \le 1+\sin 2n \le 2$.

Then, $0 \le \dfrac{1+\sin 2n}{1+\sqrt{n}} \le \dfrac{2}{1+\sqrt{n}}$....... (3 points)

So, $\lim\limits_{n\longrightarrow\infty} 0 \le \lim\limits_{n\longrightarrow\infty} \dfrac{1+\sin 2n}{1+\sqrt{n}} \le \lim\limits_{n\longrightarrow\infty} \dfrac{2}{1+\sqrt{n}}$.

Thus, $0 \le \lim\limits_{n\longrightarrow\infty} \dfrac{1+\sin 2n}{1+\sqrt{n}} \le 0$.

$\Longrightarrow \lim\limits_{n\longrightarrow\infty} \dfrac{1+\sin 2n}{1+\sqrt{n}} = 0$........ (1 point)

CONVERGES.

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Q.6. (b) . (4 points) . $a_n = \ln\sqrt{n+1} - \dfrac{1}{2}\ln n$.

Solution. $\lim\limits_{n\longrightarrow\infty} \left(\ln\sqrt{n+1} - \dfrac{1}{2}\ln n\right)$

$= \lim\limits_{n\longrightarrow\infty} \left(\dfrac{1}{2}\left[\ln(n+1) - \ln n\right]\right)$

$= \lim\limits_{n\longrightarrow\infty} \dfrac{1}{2} \ln \dfrac{(n+1)}{n}$......... (2 points)

$= \dfrac{1}{2} \ln \left[\lim\limits_{n\longrightarrow\infty} \left(\dfrac{n+1}{n}\right)\right]$

$= \dfrac{1}{2} \ln 1$

$= 0$......... (2 points)

CONVERGES.

Q.7. (8 points). For the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

(a) Verify that the integral test is applicable.

(b) Use the integral test to determine whether the series converges or diverges.

Answer to the question.

Q.7. (4 points). (a) Verify that the integral test is applicable.

Solution. (a) Let $f(x) = \frac{1}{x(1+\ln x)}$;

$D_f = (0, +\infty)$ .... (1 point)

On the interval $[1, +\infty]$, $f(x) > 0$

and so f is positive and continuous. .... (1 point)

$\ln f(x) = -\ln[x(1+\ln x)]$

$= -[\ln x + \ln(1+\ln x)]$.

So, $\frac{f'(x)}{f(x)} = -\left(\frac{1}{x} + \frac{1}{x(1+\ln x)}\right)$

$= -\left(\frac{1}{x} + f(x)\right)$.

Then $f'(x) = \underbrace{-\left(\frac{1}{x} + f(x)\right)}_{<0} \underbrace{(f(x))}_{>0} < 0$.

Hence on $[1, +\infty)$, f is decreasing.......... (2 points)

So, the integral test is applicable.

Q.7. (4 points) . (b) Use the integral test to determine whether the series converges or diverges.

Solution. $\int_1^{+\infty} f(x)\,dx = \lim\limits_{t \longrightarrow +\infty} \int_1^t \dfrac{dx}{x(1+\ln x)}$.

Set $u = \ln x$, then $du = \dfrac{dx}{x}$.

Notice: $x = 1 \Longrightarrow u = \ln 1 = 0$

So $\int_1^{+\infty} f(x)\,dx = \int_0^{+\infty} \dfrac{du}{1+u} dx$

$= [\ln(u+1)]_0^{+\infty}$

$= \lim\limits_{n \longrightarrow +\infty} \ln(u+1) - \ln 1$

$= +\infty - 0$

$= \infty$.......... (2 points)

So the integral $\int_1^{+\infty} f(x)\,dx$ diverges, ... (1 point)

Hence the series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n(1+\ln n)}$ diverges........... (1 point)

Q.8.Which (if any) of the following series converges?

If a series converges, find its sum.

If a series diverges, explain why.

(a) (5 points) . $\sum_{n=1}^{\infty} \frac{4}{n(n+2)}$.

Solution. $S_N = \sum_{n=1}^{N} \frac{4}{n(n+2)}$

$= \sum_{n=1}^{N} \left(\frac{2}{n} - \frac{2}{n+2} \right)$ .... (2 points)

$= \left(\frac{2}{1} - \frac{2}{3} \right) + \left(\frac{2}{2} - \frac{2}{4} \right) + \left(\frac{2}{3} - \frac{2}{5} \right) + ...+$

$\left(\frac{2}{N-2} - \frac{2}{N} \right) + \left(\frac{2}{N-1} - \frac{2}{N+1} \right) + \left(\frac{2}{N} - \frac{2}{N+2} \right).$

$= 2 + 1 - \frac{2}{N+1} - \frac{2}{N+2}$

$\Longrightarrow \lim_{N \longrightarrow \infty} S_N = 3$......... (2 points)

The series converges and $\sum_{n=1}^{\infty} \frac{4}{n(n+2)} = 3$......... (1 point)

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(b) (4 points) . $\sum_{n=1}^{\infty} \frac{2^{3n} \ln n}{5^n}$.

Solution. $a_n = \left(\frac{8}{5} \right)^n \ln n$......... (2 points)

$\lim_{n \longrightarrow \infty} a_n = \infty$......... (1 point)

by Divergence Test, $\sum_{n=1}^{\infty} \frac{2^{3n} \ln n}{5^n}$ diverges......... (1 point)

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(c) (5 points) . $\sum_{n=1}^{\infty}\frac{\sin(n\pi)+2^{3n}}{3^{2n}}$.

Solution.

$\sin(n\pi) = 0$ for all n,......... (1 point)

$2^{3n} = 8^n$ and $3^{2n} = 9^n$.

So, $\sum_{n=1}^{\infty}\frac{\sin(n\pi)+2^{3n}}{3^{2n}}$

$= \sum_{n=1}^{\infty}\left(\frac{8}{9}\right)^n$......... (2 points)

$= \frac{8/9}{1-8/9} = 8$......... (1 point)

Notice the Geometric series with $a = \frac{8}{9}$ and $r = \frac{8}{9} < 1$.......... (1 point)

So, the given series is convergent and its sum is equal to 8.

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Term: 082 (Saturday, February 28, $-$ Monday, June 29, 2009).

Math 102 (Calculus-II).

Major Exam-II.

Monday, May 18, 2009.

Department of Mathematics & Statistics

KFUPM.