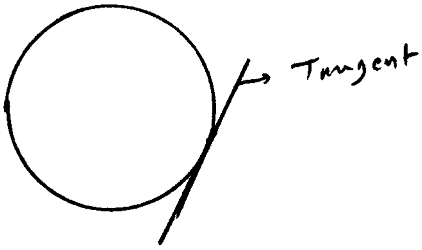
CH. 2: ✱ **The Tangent problem** ✱

Objective: To locate tangents and explore them numerically

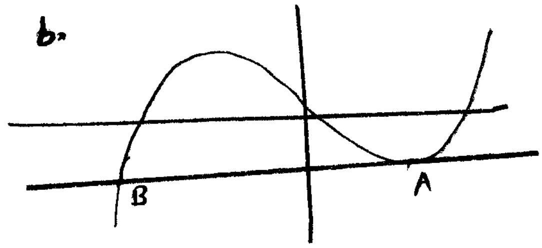
Def. Tangent: It is a line that tuchs a curve, and it should have the same direction as the curve at the point of intersection.

Ex. a.



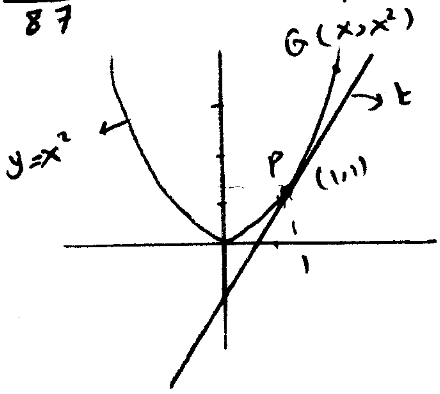
Toch it once

b.



Intersect it twice.

EX. 1 87: Find an equation of the tangent line to the parbola $y = x^2$ at $P(1,1)$.



We need to find the slope

$$m_{PQ} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{x^2 - 1}{x - 1}, \quad x \neq 1.$$

x	m_{PQ}
2	3
1.5	2.5
1.1	2.1
1.01	2.01
1.001	2.001

x	m_{PQ}
0	1
0.5	1.5
.9	1.9
.99	1.99
.999	1.999

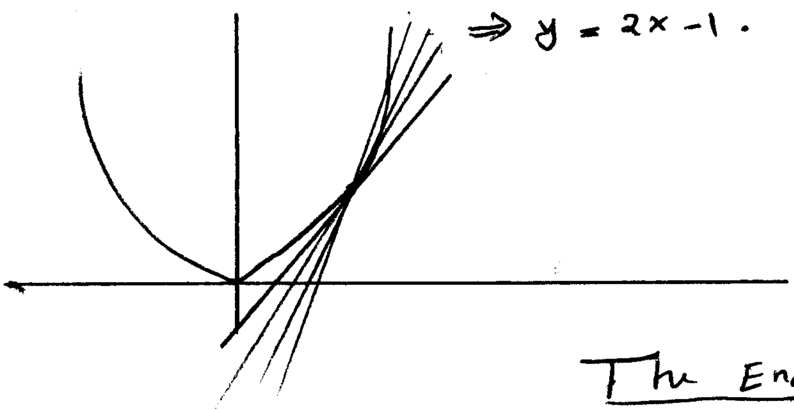
The values as x closes to 1, the slope m_{PQ} closes to 2

$\therefore m = 2$.

$$\lim_{Q \to P} m_{PQ} = m \Rightarrow \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2.$$

$\Rightarrow$ The equ. of the line is:

$$y - y_1 = m(x - x_1)$$
$$y - 1 = 2(x - 1)$$
$$\Rightarrow y = 2x - 1.$$



The End

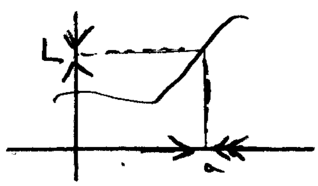
The Limit of a Function

Objectives 1. To explain the meaning of a limit, one sided limits
2. " " " " " infinite limits
3. " define the vertical asymptotes

Def The limit of the function $f(x)$ as x approaches to a is:

$$\lim_{x \to a} f(x) = L \quad , \quad x \neq a$$

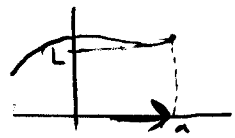
from both directions



OR, $f(x) \to L$ as $x \to a$

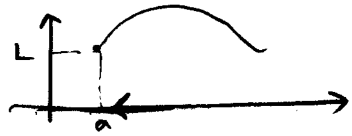
Ex: If $\lim_{x \to 3} f(x) = 6$ means As x approaches to 3, $f(x)$ approaches to 6.

Def: one-sided Limits:



1- Left-hand limit: $\lim_{x \to a^-} f(x) = L$

As x approachs a from the left, $f(x)$ approachs to L.



2. Right-hand Limit: $\lim_{x \to a^+} f(x) = L$

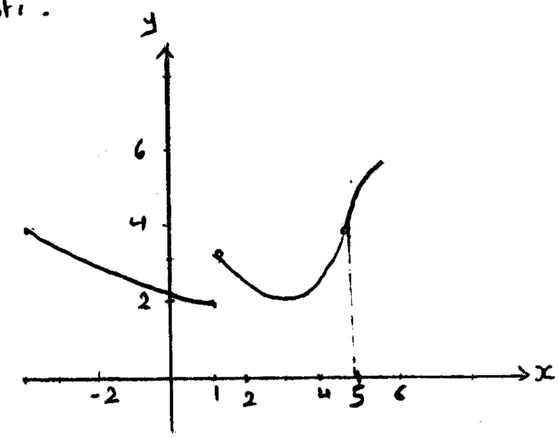
As x approachs a from the right, $f(x)$ approachs to L

So, in general,

$$\lim_{x \to a} f(x) = L \quad \text{if and only if :} \quad \lim_{x \to a^-} f(x) = L \ \text{ and } \ \lim_{x \to a^+} f(x) = L$$

Note: if $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$ then $\lim_{x \to a} f(x)$ does not exist.

Q.5, Use the graph of f(x) find each quantities if exist.
102



a. $\lim_{x \to 1^-} f(x) = 2$

b. $\lim_{x \to 1^+} f(x) = 3$

c. $\lim_{x \to 1} f(x)$ does NOT exists

because $\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$

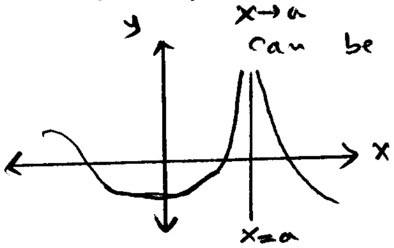
d. $\lim_{x \to 5} f(x)$: $\lim_{x \to 5^-} f(x) = 4$, $\lim_{x \to 5^+} f(x) = 4$ $\Rightarrow$ $\lim_{x \to 5} f(x) = 4$

e. $f(5)$: undefined, it does not exist.

* Infinite Limits

Let $f(x)$ be defined on both sides of a, except possible at a

then [1] $\lim_{x\to a} f(x) = \infty$ means as x approachs to a, $f(x)$ can be made arbitrarily large.

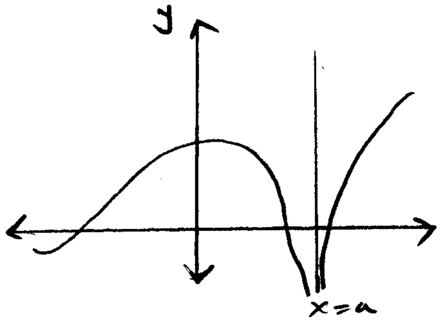


or $f(x) \to \infty$ as $x \to a$.

or $f(x)$ increases without bound as $x \to a$.

$\lim_{x\to a} f(x) = \infty$

[2] $\lim_{x\to a} f(x) = -\infty$ means as x approachs a, $f(x)$ Can be made arbitrarily large negative.

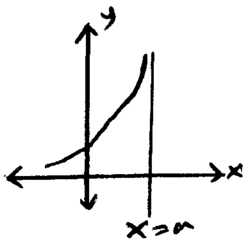


or $f(x) \to -\infty$ as $x \to a$

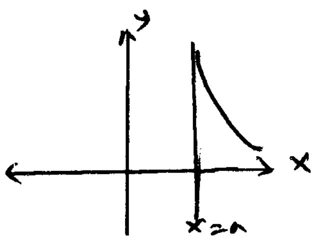
or $f(x)$ decreases without bound as $x \to a$.

$\lim_{x\to a} f(x) = -\infty$

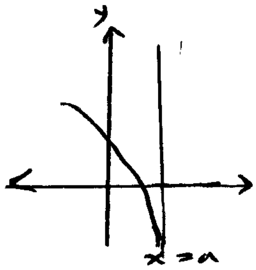
Note, similarly, we can define the one sided infinite limit.



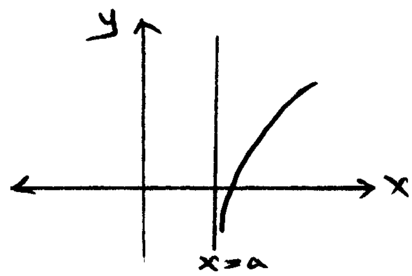
a. $\lim_{x\to a^-} f(x) = \infty$



b. $\lim_{x\to a^+} f(x) = \infty$



c. $\lim_{x\to a^-} f(x) = -\infty$



d. $\lim_{x\to a^+} f(x) = -\infty$

Def The line $x=a$ is a vertical asymptote of the curve $y=f(x)$ if at least one of the following is true:

$\lim_{x\to a} f(x) = \infty$ $\quad \lim_{x\to a^-} f(x) = \infty$ $\quad \lim_{x\to a^+} f(x) = \infty$

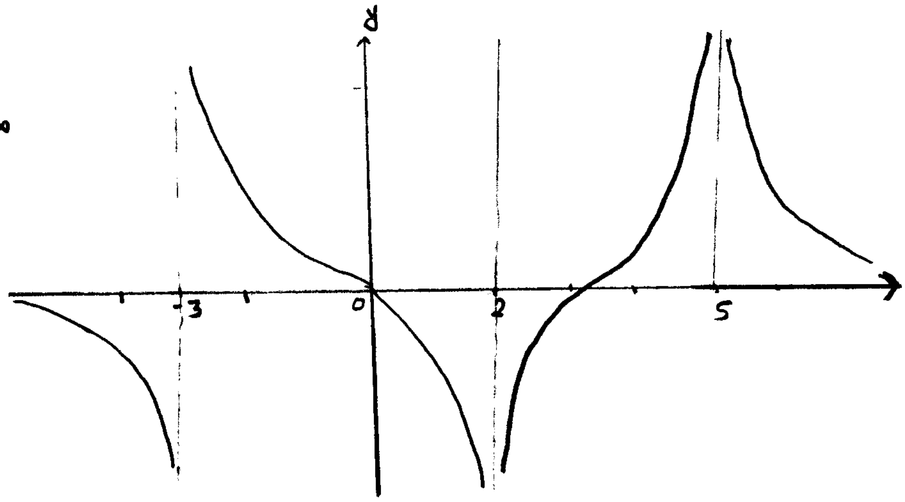
$\lim_{x\to a} f(x) = -\infty$ $\quad \lim_{x\to a^-} f(x) = -\infty$ $\quad \lim_{x\to a^+} f(x) = -\infty$.

Ex, Determine the infinite limits:

Q.24 / 103, $\lim_{x\to 5^-} \frac{6}{x-5} \to \frac{6}{0^-} = -\infty$

Q.29 / 103, $\lim_{x\to -\frac{\pi}{2}^-} \sec x = \lim_{x\to -\frac{\pi}{2}} \frac{1}{\cos x} \Rightarrow \frac{1}{0^-} = -\infty$, $\cos x < 0$ for $-\pi < x < -\frac{\pi}{2}$

$\underline{\underline{Q\ 8}}$, For the function R find the following:
$\underline{102}$



a. $\lim_{x \to 2} R(x) = ?$

$\lim_{x \to 2^-} R(x) = -\infty$, $\lim_{x \to 2^+} R(x) = -\infty$

$\therefore \lim_{x \to 2} R(x) = -\infty$

b. $\lim_{x \to 5} R(x) = ?$

$\lim_{x \to 5^-} R(x) = \infty$, $\lim_{x \to 5^+} R(x) = \infty$

$\therefore \lim_{x \to 5} R(x) = \infty$

c. $\lim_{x \to -3^-} R(x) = -\infty$

d. $\lim_{x \to -3^+} R(x) = \infty \Rightarrow \lim_{x \to -3} R(x)$ does not exist.

e. The equ. of the vertical asymptote.
Asymptotes are: $x = -3$, $x = 2$, $x = 5$

Ex. Find the vertical asymptotes of $y = \dfrac{3x+1}{x^3 - 3x^2 - 4x}$

$$y = \frac{3x+1}{x(x^2-3x-4)} = \frac{3x+1}{x(x-4)(x+1)}$$

As $x \to 0^+$, $y \to -\infty$ $\therefore$ $x = 0$ is a V.A.
As $x \to 4^+$, $y \to +\infty$ $\therefore$ $x = 4$ " "
As $x \to -1^+$, $y \to +\infty$ $\therefore$ $x = -1$ " " .

The End

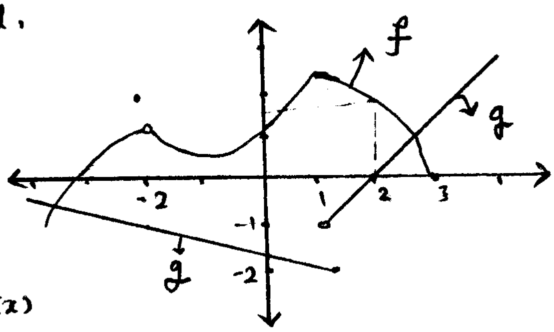
* Calculating Limits Using Limit Laws *

Objectives: 1. To introduce the limit laws
2. To find the limit of piecewise-defined function
3. To introduce the Squeeze Theorem.

Def. Limit Laws: If c is a constant and $\lim_{x\to a} f(x)$, $\lim_{x\to a} g(x)$ are exist. Then:

1. $\lim_{x\to a} [f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a} [f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. $\lim_{x\to a} [c\,f(x)] = c\lim_{x\to a} f(x)$
4. $\lim_{x\to a} [f(x)\,g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
5. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$

Ex. 1 / 105: Use the graphs of f and g to find.



a. $\lim_{x\to -2} [f(x)+5g(x)]$

$\lim_{x\to -2} f(x) = 1$, $\lim_{x\to -2} g(x) = -1$

$\therefore \lim_{x\to -2} [f(x)+5g(x)] = \lim_{x\to -2} f(x) + 5\lim_{x\to -2} g(x)$
$= 1 + 5(-1) = 1 - 5 = -4$

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b. $\lim_{x\to 1} [f(x)\,g(x)]$

$\lim_{x\to 1} f(x) = 2$, But $\lim_{x\to 1} g(x)$ d.N.E because $\lim_{x\to 1^-} g(x) = -2 \neq \lim_{x\to 1^+} g(x) = 1$

so law (4) can't be used. $\Rightarrow \lim_{x\to 1} [f(x)\cdot g(x)]$ d.N.E

Because $\lim_{x\to 1^-} [f(x)\,g(x)] = (2)(-2) = -4$
$\lim_{x\to 1^+} [f(x)\,g(x)] = (2)(1) = 2$.

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c. $\lim_{x\to 2} \dfrac{f(x)}{g(x)}$. $\lim_{x\to 2} f(x) \approx 1.4$, $\lim_{x\to 2} g(x) = 0$

$\therefore$ law (5) can't be used. because $\lim_{x\to 2^-} \dfrac{f(x)}{g(x)} = -\infty \neq \lim_{x\to 2^+} \dfrac{f(x)}{g(x)} = \infty$

$\Rightarrow$ Laws Limits: 6. $\lim_{x\to a} [f(x)]^n = [\lim_{x\to a} f(x)]^n$, n is +ve. integer

7. $\lim_{x\to a} c = c$ 8. $\lim_{x\to a} x = a$.

9. $\lim_{x\to a} x^n = a^n$, n is +ve. integer (put $f(x) = x$).

10. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$, n is +ve integer
if n is even, then $a > 0$).

11. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$, n is +ve. integer
if n is even, assume that $\lim_{x\to a} f(x) > 0$.

Ex. Evaluate the following limits

Q.3 / 112: $\lim_{x\to -2} (3x^4 + 2x^2 - x + 1)$

$$= 3\lim_{x\to -2} x^4 + 2\lim_{x\to -2} x^2 - \lim_{x\to -2} x + \lim_{x\to -2} 1$$

$$= 3(-2)^4 + 2(-2)^2 - (-2) + 1 = 48 + 8 + 2 + 1 = 59$$

Q.6 / 112: $\lim_{t\to -1} (t^2+1)^3 (t+3)^5$

$$= \lim_{t\to -1} (t^2+1)^3 \cdot \lim_{t\to -1} (t+3)^5$$

$$= \left[\lim_{t\to -1} t^2 + \lim_{t\to -1} 1\right]^3 \cdot \left[\lim_{t\to -1} t + \lim_{t\to -1} 3\right]^5$$

$$= \left[(-1)^2 + 1\right]^3 \cdot \left[(-1) + 3\right]^5 = (8)(32) = 256$$

Direct Substitution Property: If f is a poly. or rational function and a is in the domain of f then $\lim_{x\to a} f(x) = f(a)$.

Q.4 / 112: $\lim_{x\to 2} \frac{2x^2+1}{x^2+6x-4} = \frac{2(2)^2+1}{(2)^2+6(2)-4} = \frac{9}{12} = \frac{3}{4}$.

Ex. Evaluate the following limits:

Q.11 / 112: $\lim_{x\to 2} \frac{x^2+x-6}{x-2}$ Direct Substitution incorrect, $2 \notin$ dom.

$$\therefore \lim_{x\to 2} \frac{x^2+x-6}{x-2} = \lim_{x\to 2} \frac{(x+3)\cancel{(x-2)}}{\cancel{(x-2)}} = \lim_{x\to 2} (x+3) = 5$$

Q.20 / 112: $\lim_{h\to 0} \frac{(2+h)^3 - 8}{h} = \lim_{h\to 0} \frac{8 + 12h + 6h^2 + h^3 - 8}{h}$

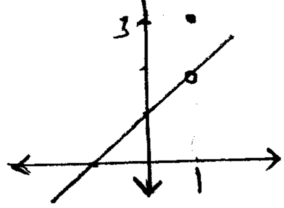
$$= \lim_{h\to 0} \frac{12h + 6h^2 + h^3}{h} = \lim_{h\to 0} \frac{\cancel{h}(12 + 6h + h^2)}{\cancel{h}}$$

$$= \lim_{h\to 0} (12 + 6h + h^2) = 12 + 0 + 0 = 12 .$$

Q.30 / 112: $\lim_{x\to 1} \frac{\sqrt{x} - x^2}{1-\sqrt{x}} \cdot \frac{1+\sqrt{x}}{1+\sqrt{x}} = \lim_{x\to 1} \frac{\sqrt{x} + x - x^2 - \sqrt{x}\,x^2}{1-x} = \lim_{x\to 1} \frac{(\sqrt{x} - \sqrt{x}\,x^2) + (x - x^2)}{1-x}$

$$= \lim_{x\to 1} \frac{\sqrt{x}(1-x)(1+x) + x(1-x)}{1-x} = \lim_{x\to 1} \frac{\cancel{(1-x)}\left[\sqrt{x}(1+x) + x\right]}{\cancel{(1-x)}} = 1(1+1)+1 = 2+1 = 3$$

Ex. 4 / 108: Find $\lim_{x\to 1} g(x)$, where $g(x) = \begin{cases} x+1, & x \neq 1 \\ \pi, & x = 1 \end{cases}$



$\lim_{x\to 1} g(x) = \lim_{x\to 1} (x+1) = 1+1 = 2$, but $g(1) = 2$

Thm. 1: $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$

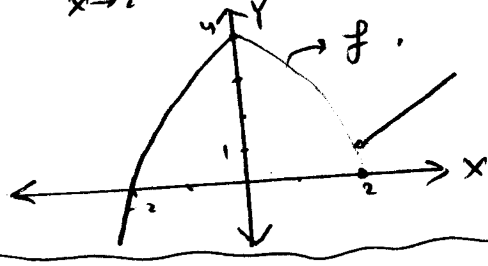
Q. 46 / 113: Let $f(x) = \begin{cases} 4-x^2 & \text{if } x \le 2 \\ x-1 & \text{if } x > 2 \end{cases}$ Find:

a. $\lim_{x\to 2^-} f(x)$ & $\lim_{x\to 2^+} f(x)$ (i) $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (4-x^2) = 4-(2)^2 = 0$

(ii) $\lim_{x\to 2^+} f(x) = \lim_{x\to 2} (x-1) = 2-1 = 1$

b. Does $\lim_{x\to 2} f(x)$ exists? No because $\lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$

c. Sketch f.



Q. 40 / 112: $\lim_{x\to -4^-} \frac{|x+4|}{x+4}$, $|x+4| = \begin{cases} x+4, & x \ge -4 \\ -(x+4), & x < -4 \end{cases} \Rightarrow \frac{|x+4|}{x+4} = \begin{cases} \frac{x+4}{x+4} & \text{if } x > -4 \\ \frac{-(x+4)}{x+4} & \text{if } x < -4 \end{cases}$

$\frac{|x+4|}{x+4} = \begin{cases} 1 & \text{if } x > -4 \\ -1 & \text{if } x < -4 \end{cases}$

$\therefore \lim_{x\to -4^-} \frac{|x+4|}{x+4} = \lim_{x\to -4^-} -1 = -1$.

Q. 44 / 113: $\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{|x|}\right)$, For $x \to 0^+ > 0 \Rightarrow |x| = \begin{cases} x, & x \ge 0 \\ -x, & x < 0 \end{cases}$

$\Rightarrow |x| = x$

$= \lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{x}\right) = \lim_{x\to 0^+} (0) = 0$

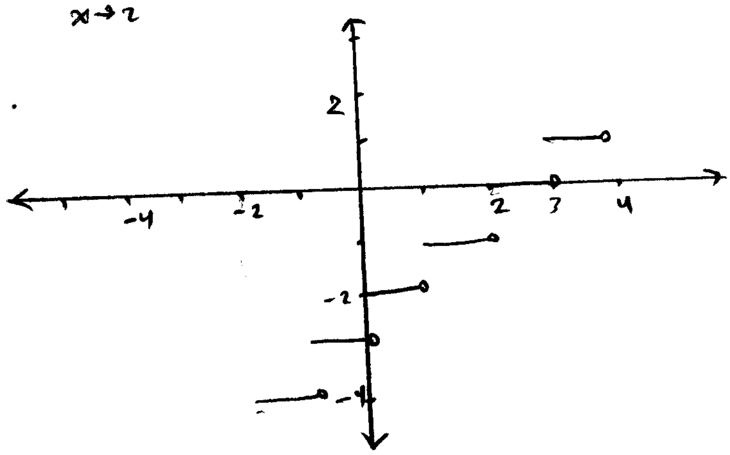
Ex: If $[\![\ ]\!]$ denotes the gratest integer function, $f(x) = [\![x+2]\!]$ find,

a. $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} [\![x-2]\!] = -1$ $\Rightarrow$ $\lim_{x\to 2} f(x)$ does Not exist.

b. $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} [\![x-2]\!] = 0$.

c. $\lim_{x\to 3.1} f(x) = \lim_{x\to 3.1} [\![x-2]\!] = 1$.

d. Sketch the graph of f.



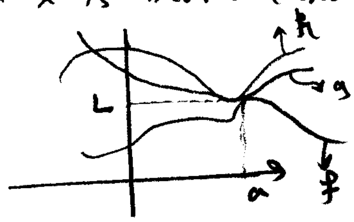
Thm 2. If $f(x) \le g(x)$ when x is near a (except possibly at a) and $\lim_{x\to a} f(x)$, $\lim_{x\to a} g(x)$ are exist, then

$$\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$$

Thm 3. The squeeze Theorem: If $f(x) \le g(x) \le h(x)$ when x is near a (except possibly at a) and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$

then:

$$\lim_{x\to a} g(x) = L$$



It is also called the Sandwich Theorem or Pinching theorem.

Q.34 112: Show that $\lim_{x\to 0} \sqrt{x^3+x^2}\,\sin\frac{\pi}{x} = 0$

$-1 \le \sin\frac{\pi}{x} \le 1$ multiply by $\sqrt{x^3+x^2}$

$-\sqrt{x^3+x^2} \le \sqrt{x^3+x^2}\,\sin\frac{\pi}{x} \le \sqrt{x^3+x^2}$

$\lim_{x\to 0} -\sqrt{x^3+x^2} = 0$ and $\lim_{x\to 0} \sqrt{x^3+x^2} = 0$

$\Rightarrow$ by the Squeeze Theorem $\lim_{x\to 0} \sqrt{x^3+x^2}\,\sin\frac{\pi}{x} = 0$

Q.35 112: If $1 \le f(x) \le x^2+2x+2$ for all x find $\lim_{x\to -1} f(x)$

$\lim_{x\to -1} 1 = 1$, $\lim_{x\to -1} (x^2+2x+2) = 1-2+2 = 1$

$\therefore$ by the Squeeze Theorem $\lim_{x\to -1} f(x) = 1$.

Q.50 113: Let $f(x) = x - [\![x]\!]$

$[\![x]\!] = n$ iff $n \le x < n+1$

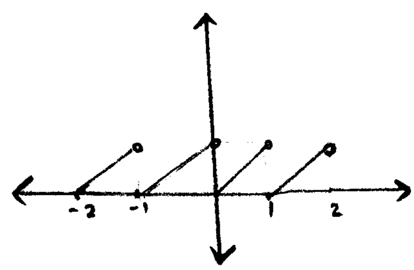
a. Sketch f.

$-2 \le x < -1 \to f = x+2$

$-1 \le x < 0 \to f = x+1$

$0 \le x < 1 \to f = x$

$1 \le x < 2 \to f = x-1$



b. If n is an integer find

(i) $\lim_{x\to n^-} f(x) = \lim_{x\to n^-} (x - [\![x]\!])$

$= \lim_{x\to n^-} x - \lim_{x\to n^-} [\![x]\!] = n-(n-1) = 1$

(ii) $\lim_{x\to n^+} f(x) = \lim_{x\to n^+} x - \lim_{x\to n^+} [\![x]\!] = n-n = 0$ $\Rightarrow \lim_{x\to n} f(x)$ d.N.E

c. For what values of a does $\lim_{x\to a} f(x)$ exist?

$\lim_{x\to a} f(x)$ exists if and only if a is not an integer

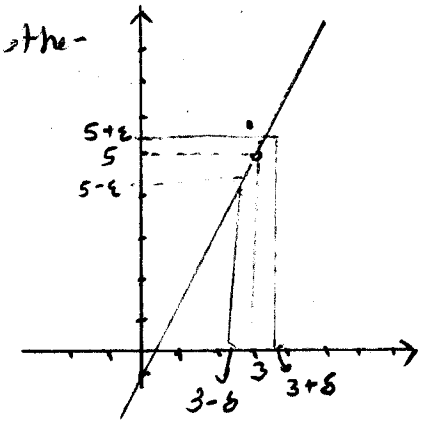
The End.

Sec. 2.4

The Precise Definition of a Limit

Objectives:
1. To define the limit using ϵ-δ definition
2. " consider the left-hand limit
3. " " " right-hand limit

- Consider the function $f(x) = \begin{cases} 2x-1, & \text{if } x \neq 3 \\ 6, & \text{if } x = 3 \end{cases}$ then



$\lim\limits_{x\to 3} f(x) = \lim\limits_{x \to 3}(2x-1) = 5$

Questions: How close to 3 does x have to be, so that $f(x)$ differs from 5 by less than .1?

Distance between x & 3 is $|x-3|$

" " $f(x)$ & 5 $= |f(x)-5|$

We need $|f(x)-5| < .1$ if $|x-3| < \delta$ but $x \neq 3 \Rightarrow 0 < |x-3| < \delta$

If $|x-3| < \delta = \frac{.1}{2} = .05$ then:

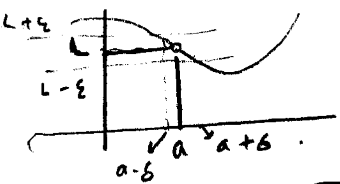
$|f(x)-5| = |2x-1-5| = |2x-6| = 2|x-3| < .1 \Rightarrow |x-3| < \frac{.1}{2} = .05$ tolerance ↑

So $|f(x)-5| < .1$ if $0 < |x-3| < .05$. **Note** .1 is called an error

Def: Let f be a func. defined on an open interval contains a, except possible at a, then $\lim\limits_{x\to a} f(x) = L$

if for every $\varepsilon > 0$, there is a number $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x-a| < \delta$.

OR if $0 < |x-a| < \delta$ then $|f(x)-L| < \epsilon$



Ex: Use ε-δ definition to prove the given limit:

Q-18 / 123: $\lim\limits_{x\to 4}(7-3x) = -5$

Let $\epsilon > 0$, then there is $\delta > 0$ such that

$|f(x) - (-5)| < \epsilon$ whenever $0 < |x-4| <$

$|7-3x+5| < \epsilon$ " $0 < |x-4| <$

$|12-3x| < \epsilon$ " $0 < |x-4| < \delta$

$|(-3)(x-4)| < \epsilon$ " "

$3|x-4| < \epsilon$ " "

$\Rightarrow |x-4| < \frac{\epsilon}{3}$ " " $\Rightarrow$ choose $\delta = \frac{\epsilon}{3}$.

Showing that δ works.

Given $\epsilon > 0$, $\delta = \frac{\epsilon}{3}$. If $0 < |x-4| < \delta$, then

$|7-3x-(-5)| = |12-3x| = 3|x-4| < 3\delta = \epsilon$.

Def. of left-Hand limit: $\lim_{x \to a^-} f(x) = L$

if for every $\epsilon > 0$ there is a number $\delta > 0$ such that
$|f(x) - L| < \epsilon$ whenever $a - \delta < x < a$.

Def. of Right-Hand limit: $\lim_{x \to a^+} f(x)$

if for every $\epsilon > 0$, there is a number $\delta > 0$ such that
$|f(x) - L| < \epsilon$ whenever $a < x < a + \delta$.

Q.20 / 123 Recite: $\lim_{x \to 6} \left(\frac{x}{4} + 3\right) = \frac{9}{2}$.

Let $\epsilon > 0$, then there is $\delta > 0$ such that
$|f(x) - \frac{9}{2}| < \epsilon$ whenever $0 < |x - 6| < \delta$.

But $|f(x) - \frac{9}{2}| = |\frac{x}{4} + 3 - \frac{9}{2}| = |\frac{x}{4} - \frac{3}{2}| = |\frac{x-6}{4}| = \frac{1}{4}|x-6|$

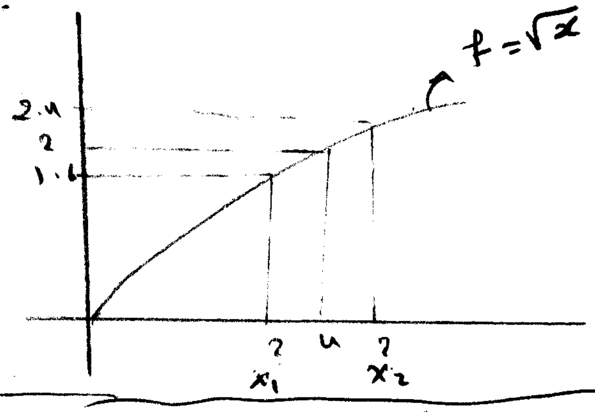
$\Rightarrow \frac{1}{4}|x-6| < \epsilon$ whenever $0 < |x-6| < \delta$
$\Rightarrow |x-6| < 4\epsilon$ " "
$\therefore$ choose $\delta = 4\epsilon$. or any smaller +ve number.

Ex. 3 / 118: Prove that $\lim_{x \to 0^+} \sqrt{x} = 0$

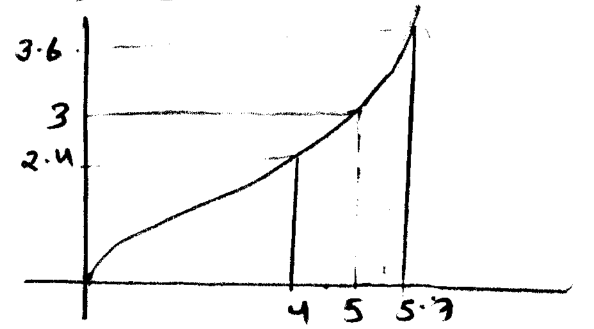
Let $\epsilon > 0$ then, there is $\delta > 0$ such that
$|\sqrt{x} - 0| < \epsilon$ whenever $0 < (x - 0) < 0 + \delta$
$|\sqrt{x}| < \epsilon$ " $0 < x < \delta$
$\sqrt{x} < \epsilon$ " "
$x < \epsilon^2$ " " $\Rightarrow$ choose $\delta = \epsilon^2$.

Q.5 / 122: Use the graph of $f(x) = \sqrt{x}$ to find δ such that
$|\sqrt{x} - 2| < .4$ whenever $|x - 4| < \delta$.

$x_1 = (1.6)^2 = 2.56$
$x_2 = (2.4)^2 = 5.76$
$\Rightarrow |a - x_1| = |4 - 2.56| = 1.44$
$|a - x_2| = |4 - 5.76| = 1.76$
$\therefore$ Choose $\delta = 1.44$



Q.4 / 122 Recite: Use the graph of f to find δ such that
$|f(x) - 3| < .6$ whenever $0 < |x - 5| < \delta$
$\delta_1 = |a - 4| = |5 - 4| = 1$
$\delta_2 = |a - 5.7| = |5 - 5.7| = .7$
$\therefore$ choose $\delta = \min(1, .7) = .7$.



The End

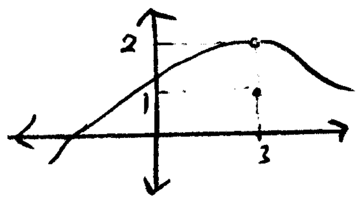
* Continuity *

Objectives: 1. To define a continuity of $f(x)$ at $x=a$, left-conts and right conts.
2. " " " on $[a,b]$, and for $(f\circ g)(x)$
3. " introduce the Intermediate Value Theorem (I.M.T.)

Def. A function f is continuous at a if $\lim_{x\to a} f(x) = f(a)$

Note. The definition requires three things:
1. $f(a)$ is defined (or $a \in dom(f)$)
2. $\lim_{x\to a} f(x)$ exists
3. $\lim_{x\to a} f(x) = f(a)$

Ex. Consider the following graph of f, Does f continuous at $x=3$



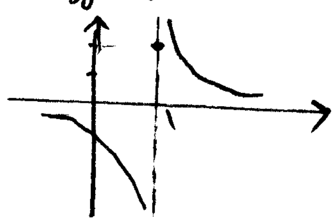
1. $f(3)$ is defined, $f(3)=1$
2. $\lim_{x\to 3} f(x) = 2$
3. $\lim_{x\to 3} f(x) \neq f(3)$ $\therefore$ f is discountinnous at $x=3$

Q.12 / 133, $g(x) = \dfrac{x+1}{2x^2-1}$, $a=4$

1. $g(4) = \frac{5}{31}$ 2. $\lim_{x\to 4} g(x) = \lim_{x\to 4} \dfrac{x+1}{2x^2-1} = \dfrac{5}{31}$ 3. $\lim_{x\to 4} g(x) = g(4)$

$\Rightarrow$ $g(x)$ is conts. at $x=4$

Q.16 / 133 Explain why f not conts. at a : $f(x) = \begin{cases} \dfrac{1}{x-1} & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$



1. $f(1) = 2$ defind.
2. $\lim_{x\to 1^-} f(x) = -\infty$, $\lim_{x\to 1^+} f(x) = +\infty$ $\Rightarrow$ Li

$\Rightarrow$ $\lim_{x\to 1} f(x)$ d.N.E.

$\therefore$ $f(x)$ is not conts. at $x=1$ because $\lim_{x\to 1} f(x)$ d.N.E.

Ex. (6. Exam) : Find all values of A and B which will make f continuous (8 pts).

$$f(x) = \begin{cases} x^2 - A & \text{if } x<1 \\ A+Bx & \text{if } 1 \le x \le 2 \\ B - x^2 & \text{if } 2<x \end{cases}$$

(1) $f(x)$ is conts. at $x=1$ $\Rightarrow$ $\lim_{x\to 1} f(x) = f(1)$ $\Rightarrow$ $1-A = A+B$ $\Rightarrow$ $2A+B = 1$ — ①

(2) " " " " $x=2$ $\Rightarrow$ $\lim_{x\to 2} f(x) = f(2)$ $\Rightarrow$ and $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$

$A+2B = B-4$
$\Rightarrow A = -B-4$ —— ②

Substitute A from (2) in ① $\Rightarrow$ $2(-B-4)+B = 1$
$\Rightarrow -B = 1+8 = 9$ $\Rightarrow$ $B=-9$ $\therefore$ $A = -(-9)-4 = 5$.

Def. 1. A function f is conts. from the right at a if
$$\lim_{x\to a^+} f(x) = f(a)$$

2. A function f is conts. from the left at a if
$$\lim_{x\to a^-} f(x) = f(a)$$

Ex. Let $f(x) = [\![x]\!]$, and let n be an integer number, Discuss the continuity of f from the right and left of n.

(i) Right-conts.

1. $f(n) = [\![n]\!] = n$ defined
2. $\lim_{x\to n^+} f(x) = [\![n^+]\!] = n$
3. $\lim_{x\to n^+} f(x) = f(n)$

$\therefore$ f is conts. from the right.

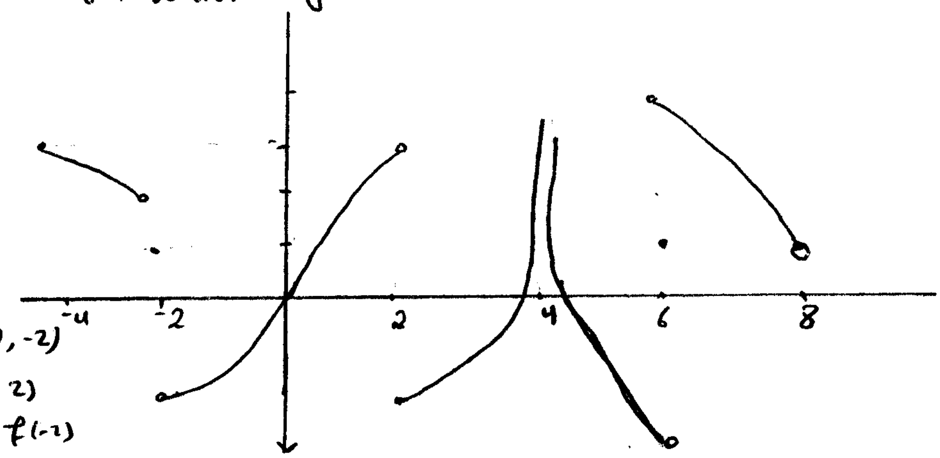
(ii) Left-conts.

1. $f(x)$ is defined at n, $f(n) = [\![n]\!] = n$.
2. $\lim_{x\to n^-} f(x) = [\![n^-]\!] = n-1$
3. $\lim_{x\to n^-} f(x) \neq f(n)$

$\therefore$ f is discontinuous from the left at $x=n$.

Def. A function f is conts on $[a,b]$ if it is conts at every point in the interval, if f is defined on one side of an end point, then it must be conts either from left or right.

Q-4 State the intervals on which f is conts.
133 Not now.



At $x=-4$

$\lim_{x\to -4} f(x) = f(-4)$

$\lim_{x\to -2} f(x) \neq f(-2)$

1. $\therefore$ $f(x)$ conts on $[-4,-2)$
2. f is conts. on $(-2,2)$
   because $\lim_{x\to -2^+} f(x) \neq f(-2)$
   $\lim_{x\to 2} f(x) \neq f(2)$
3. f is conts on $[2,4)$ but at $x=4$ undefined
4. " " " $(4,6)$ because $\lim_{x\to 6^-} f(x) \neq f(6)$
5. " " " $(6,8)$ " $\lim_{x\to 6^+} f(x) \neq f(6)$, and $f(8)$ undefined.

Ex. 4 Show that $f(x) = 1-\sqrt{1-x^2}$ is conts on $[-1,1]$
126

1. For $-1<a<1$, $\lim_{x\to a} f(x) = \lim_{x\to a}(1-\sqrt{1-x^2}) = 1-\sqrt{1-a^2} = f(a)$
2. $\lim_{x\to -1^+} f(x) = 1-\sqrt{1-1} = 1-0 = 1 = f(-1)$
3. $\lim_{x\to 1^-} f(x) = 1-\sqrt{1-1} = 1-0 = 1 = f(1)$

$\therefore$ $f(x)$ is conts. on $[-1,1]$

Thm (4): If f and g are conts. at a and c is a constant then the following functions are conts. at a:

1. $f+g$ 2. $f-g$ 3. cf 4. fg 5. $\frac{f}{g}$ if $g(a) \neq 0$

Q.9 / 133: If f and g are conts. with $f(3) = 5$, $\lim_{x\to 3} [2f(x) - g(x)] = 4$, find $g(3)$

$\lim_{x\to 3} [2f(x) - g(x)] = 2\lim_{x\to 3} f(x) - \lim_{x\to 3} g(x)$ because f & g are conts $\Rightarrow$ Their limits are exist.

but $\lim_{x\to 3} f(x) = f(3) = 5$

$\Rightarrow 2(5) - \lim_{x\to 3} g(x) = 4 \Rightarrow \lim_{x\to 3} g(x) = 10 - 4 = 6 = g(3)$

Thm(5): 1. Any poly. is conts. everywhere i.e: on $\mathbb{R} = (-\infty, +\infty)$

2. Any rational function is conts. whenever it is defined i.e on its domain

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Q-31 / 134: Use continuity to evaluate the limit: $\lim_{x\to 4} \frac{5+\sqrt{x}}{\sqrt{5+x}}$

Let $f = \frac{5+\sqrt{x}}{\sqrt{5+x}}$, Dom. $f(x)$: $(0, \infty)$, $4 \in \text{dom}(f)$

$\Rightarrow \lim_{x\to 4} f(x) = f(4) = \frac{5+\sqrt{4}}{\sqrt{5+4}} = \frac{7}{3}$.

Thm All of the following are conts. at every number in their domain.

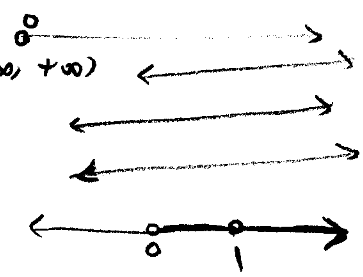
1. polynomials 2. Rational functions 3. Root functions
4. Trigonometric functions 5. Inverse trigonometric functions 6. Exponential functions
7. Logarithmic functions.

Ex. 6 / 129: where is $f(x) = \frac{\ln x + \tan^{-1}(x)}{x^2 - 1}$ continuous?

1. $\ln x$ is continuous for all $x > 0$
2. $\tan^{-1}(x)$ " " " all real numbers $\mathbb{R} = (-\infty, +\infty)$
3. $x^2 - 1$ " " " " " "

but $f(x)$ is undefined on $-1, 1$

$\therefore f(x)$ is conts on $(0,1) \cup (1, \infty)$



Thm: If f is conts at b and $\lim_{x\to a} g(x) = b$, then $\lim_{x\to a} f(g(x)) = f(b)$

or $\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x))$

Q.32 / 134: $\lim_{x\to \pi} \sin(x + \sin x)$

x is conts on $\mathbb{R}$, $\sin x$ is conts on $\mathbb{R}$

$\therefore \lim_{x\to \pi} \sin(x + \sin \pi) = \sin(\lim_{x\to \pi}(x + \sin x))$

$= \sin(\pi + 0) = 0$

Thm: If g is conts at a and f is conts. at $g(a)$ then the Composition $(f \circ g)(x) = f(g(x))$ is conts. at a.

Q.26 / 134 Explain why the func. is conts. on its domain?

$$F(x) = \sin^{-1}(x^2-1)$$

1. x^2-1 is a poly. conts everywhere, on $\mathbb{R} = (-\infty, \infty)$
2. $\sin^{-1} x$ " conts on its domain $[-1,1]$.

$\therefore$ $F(x)$ is conts. on its domain.

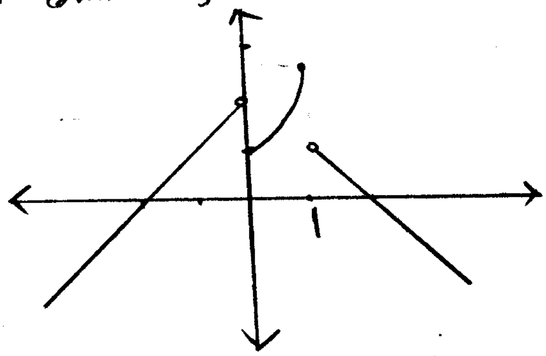
Dom$(F(x))$: $-1 \le x^2-1 \le 1$

$0 \le x^2 \le 2 \Rightarrow |x| \le \sqrt{2} \Rightarrow -\sqrt{2} \le x \le \sqrt{2}$.

$\therefore$ $F(x)$ is conts. on $[-\sqrt{2}, \sqrt{2}]$.

Q.39, / 134 Find all numbers at which f is disconts. or conts. → right or left -, Sketch f -

$$f(x) = \begin{cases} x+2 & \text{if } x<0 \\ e^x & \text{if } 0 \le x \le 1 \\ 2-x & \text{if } x>1 \end{cases}$$



1. $f(x)$ is conts. on $(-\infty, 0)$ → poly
2. " " " " $(0,1)$ → exp.
3. " " " " $(1,\infty)$ → poly.
4. At $x=0$ a. $0 \in \text{dom}(f)$ b. $\lim\limits_{x\to 0^-} f(x) = 2$, $\lim\limits_{x\to 0^+} f(x) = e^0 = 1$

$\therefore$ f is discont. at $x=0$

but $f(0)=1 \Rightarrow f$ is conts. from the right at 0

5. At $x=1$

a. $f(1) = e^1 = e$ defined b. $\lim\limits_{x\to 1^-} f(x) = e^1 = e$, $\lim\limits_{x\to 1^+} f(x) = 2-1 = 1$

$\therefore$ $\lim\limits_{x\to 1} f(x)$ d.N.E $\Rightarrow f$ is disc. at $x=1$

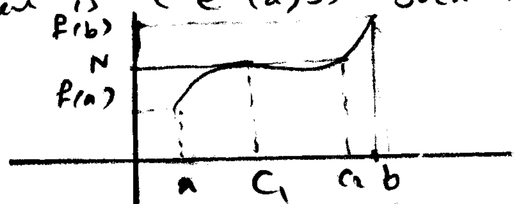
but because $\lim\limits_{x\to 1^-} f(x) = f(1)$ $\therefore$ f is conts. from the left at $x=1$

The Intermediate Value Theorem (I.V.T): If $f(x)$ is a function conts. on $[a,b]$, and let N be a number between $f(a)$ & $f(b)$ where $f(a) \ne f(b)$ then there is exist a number $c \in (a,b)$ such that $f(c) = N$

Note, I.V.T: 1. $f(x)$ conts on $[a,b]$

2. $f(a) \ne f(b)$

Conclusion: Then is $c \in (a,b)$ such that $f(c) = N$ between $f(a), f(b)$



Q.48 / 134, Use the I.V.T to show that there is a root on (a,b) of the equ. $\sqrt[3]{x} = 1-x$, $(0,1)$

Let $f(x) = \sqrt[3]{x} - 1 + x$, $x \in [0,1]$

1. $f(x)$ is Const. on $[0,1]$
2. $f(0) = 1$ & $f(1) = 1-1+1 = 1 \Rightarrow f(0) \cdot f(1) < 0 \Rightarrow -1 < \overset{N}{\downarrow}{0} < 1$.

$\therefore$ There is at least $c \in (0,1)$ such that $f(c)=0$ between $f(0) = -1$ and $f(1) = 1 \Rightarrow -1 < f(c)=0 < 1$.

The End

Q.45 / 134

If $f(x) = x^3 - x^2 + x$, show that there is a number c such that $f(c) = 10$.

Sec. 2.6

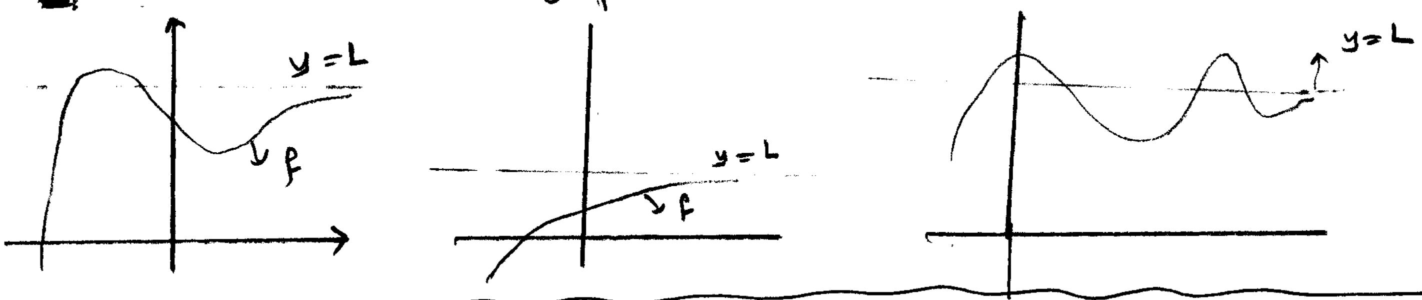
Limits at Infinity; Horizontal Asymptotes

Objectives:
1. To define limits at infinity
2. " " vertical asymptotes (V.A)
3. " introduce infinite limits at infinity.

Def: Let f be a function defined on (a,∞), then $\lim\limits_{x\to\infty} f(x) = L$ means that $f(x)$ approaches to L as x gets sufficiently large.

OR: $f(x) \to L$ as $x \to \infty$

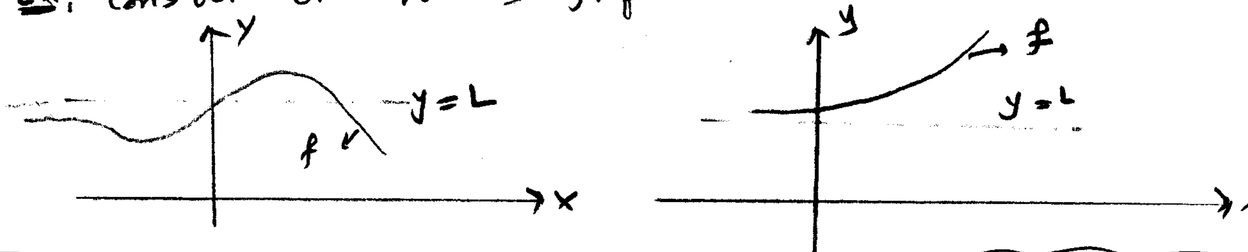
Ex: Consider the following graphs:



Def: Let f be defined on $(-\infty, a)$, then $\lim\limits_{x\to-\infty} f(x) = L$ means that $f(x)$ approaches to L as x gets sufficiently large negative.

OR: $f(x) \to L$ as $x \to -\infty$

Ex: Consider the following graphs:

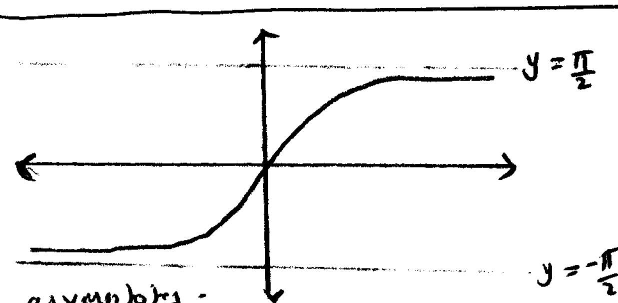


Def: The line $y = L$ is called a horizontal asymptote of the curve $y = f(x)$ if either $\lim\limits_{x\to\infty} f(x) = L$ or $\lim\limits_{x\to-\infty} f(x) = L$.

Note: For $y = \tan^{-1} x$ we have:

$$\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}, \quad \lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2}$$

$\Rightarrow$ Both lines $y = \pm\frac{\pi}{2}$ are horizontal asymptotes.



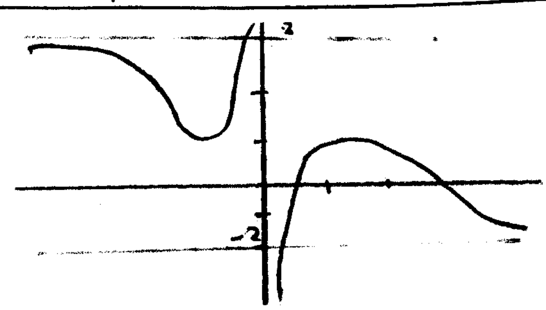
Ex: Consider the following graph: Find:

a. $\lim\limits_{x\to 0^-} f(x) = \infty$ b. $\lim\limits_{x\to 0^+} f(x) = -\infty$

c. $\lim\limits_{x\to +\infty} f(x) = -2$ d. $\lim\limits_{x\to -\infty} f(x) = 3$

e. The equ. of asymptotes: H.A: $y = -2$, $y = 3$

V.A: $x = 0$



Note: Most of the limit laws (1 − 8) in sec. 2.3 also hold for limits at infinity. (except 9 & 10 $\lim_{x\to a} x^n$, $\lim_{x\to a} \sqrt[n]{x}$

For 6 and 11 we have the following theorem:

Thm 1. If $r>0$ is a rational number, then

$$\lim_{x\to\infty} \frac{1}{x^r} = 0$$

2. If x^r is defined for all x, then $\lim_{x\to-\infty} \frac{1}{x^r} = 0$

Ex, Find the limit:

Q.14 / 147: $\lim_{x\to\infty} \frac{3x+5}{x-4}$

NOTE: To evaluate the limit at infinity of any rational function divide both the numerator and denominator by the highest power of x in the denominator.

$$\lim_{x\to\infty} \frac{3x+5}{x-4} = \lim_{x\to\infty} \frac{\frac{3x}{x}+\frac{5}{x}}{\frac{x}{x}-\frac{4}{x}} = \frac{3+\lim_{x\to\infty}\frac{5}{x}}{1-\lim_{x\to\infty}\frac{4}{x}} = \frac{3+0}{1-0} = 3.$$

$y=3$ is a H.A.

Q.20 / 147: $\lim_{x\to\infty} \frac{x+2}{\sqrt{9x^2+2}}$, $\sqrt{x^2}=|x|$ but as $x\to\infty$, $\sqrt{x^2}=x$.

$$\lim_{x\to\infty} \frac{x+2}{\sqrt{9x^2+2}} = \lim_{x\to\infty} \frac{\frac{x}{x}+\frac{2}{x}}{\sqrt{\frac{9x^2}{x^2}+\frac{2}{x^2}}} = \lim_{x\to\infty} \frac{1+2/x}{\sqrt{9+\frac{2}{x^2}}} = \frac{1+0}{\sqrt{9+0}} = \frac{1}{3}.$$

$\Rightarrow y=\frac{1}{3}$ is a H.A.

Q.24 / 147: $\lim_{x\to-\infty} (x+\sqrt{x^2+2x})$

$$\lim_{x\to-\infty} (x+\sqrt{x^2+2x}) = \lim_{x\to-\infty} x+\sqrt{x^2+2x} \cdot \frac{x-\sqrt{x^2+2x}}{x-\sqrt{x^2+2x}} = \lim_{x\to-\infty} \frac{x^2-(x^2+2x)}{x-\sqrt{x^2+2x}}$$

As $x\to-\infty$, $\sqrt{x^2}=|x|=-x$

$$= \lim_{x\to-\infty} \frac{-2x}{x-\sqrt{x^2+2x}} = \lim_{x\to-\infty} \frac{\frac{-2x}{x}}{\frac{x}{|x|}-\sqrt{\frac{x^2+2x}{x^2}}} = \lim_{x\to-\infty} \frac{-2}{1+\sqrt{1+\frac{2}{x}}} = \frac{-2}{1+1} = -1$$

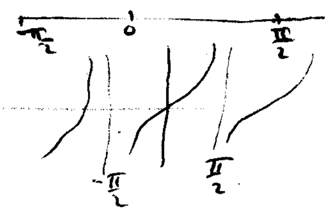
$-\sqrt{1+\frac{2}{x}}$

not now

Q.28 / 147: $\lim_{x\to-\infty} \sqrt[3]{x} = -\infty$ } **NOTE**: $\lim_{x\to-\infty} e^x = 0$

Q.34 / 147: $\lim_{x\to\frac{\pi}{2}^+} e^{\tan x} = ?$ As $x\to\frac{\pi}{2}^+$, $\tan x\to-\infty$

$$\Rightarrow \lim_{x\to\frac{\pi}{2}^+} e^{\tan x} = \lim_{y\to-\infty} e^y = 0, \quad y=\tan x$$



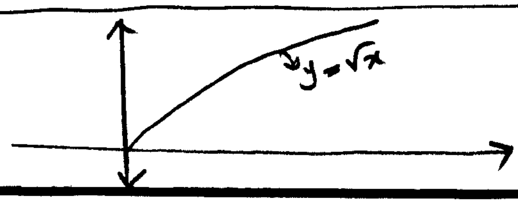
* Infinite Limits at infinity

1. $\lim_{x\to\infty} f(x) = \infty$ means $f(x)$ gets large without bound as x gets large without bound.

Similarly we can define:

2. $\lim_{x\to\infty} f(x) = -\infty$ 3. $\lim_{x\to-\infty} f(x) = \infty$ 4. $\lim_{x\to-\infty} f(x) = -\infty$

Q.27 / 147: $\lim_{x\to\infty} \sqrt{x} = \infty$



Ex. 9, 142: Find $\lim_{x\to\infty} (x^2 - x)$

Note It is not correct that: $\lim_{x\to\infty} (x^2-x) = \lim_{x\to\infty} x^2 - \lim_{x\to\infty} x = \infty - \infty$?
which can't be defined.

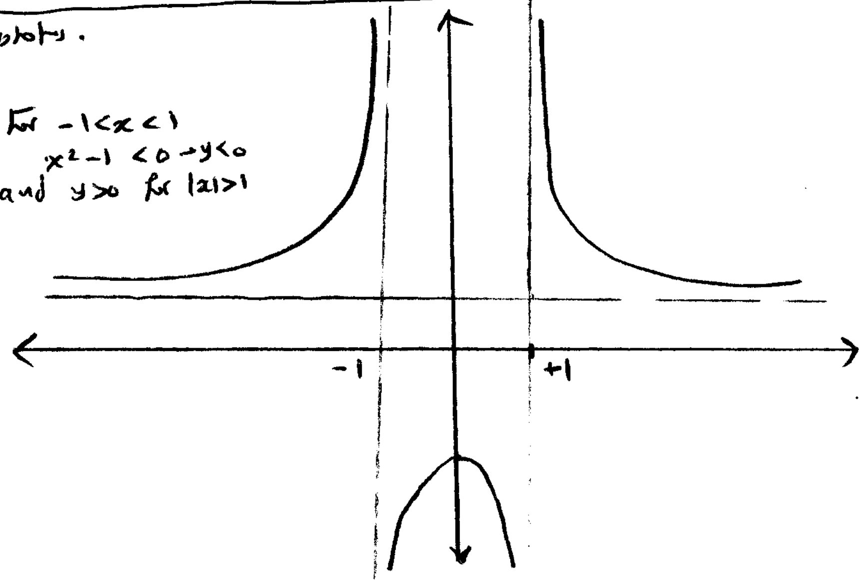
So: $\lim_{x\to\infty} x^2 - x = \lim_{x\to\infty} x(x-1) = \infty$.

Q.30, 147: $\lim_{x\to\infty} \dfrac{x^3-2x+3}{5-2x^2} = \lim_{x\to\infty} \dfrac{\frac{x^3}{x^2} - \frac{2x}{x^2} + \frac{3}{x^2}}{\frac{5}{x^2} - \frac{2x^2}{x^2}} = \lim_{x\to\infty} \dfrac{x - 2/x + 3/x^2}{\frac{5}{x^2} - 2}$
$= -\infty$.

Ex: Find the H. & V. Asymptots.

Q.38, 147: $y = \dfrac{x^2+4}{x^2-1}$.

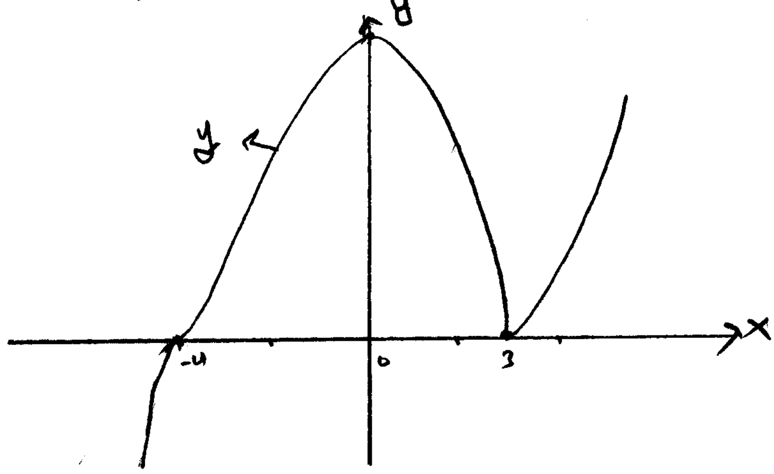
For $-1<x<1$
$x^2-1<0 \to y<0$
and $y>0$ for $|x|>1$



1. $\lim_{x\to 1^-} y = -\infty$, $\lim_{x\to 1^+} y = \infty$
   $\Rightarrow x=1$ is a V.A.
2. $\lim_{x\to -1^-} y = +\infty$, $\lim_{x\to -1^+} y = -\infty$
   $\Rightarrow x=-1$ is a V.A.
3. **H.A.**
   $\lim_{x\to\infty} y = 1$, $\lim_{x\to -\infty} y = 1$
   $\Rightarrow y=1$ is a H.A.

Q.47, 148: Find $\lim_{x\to\pm\infty} f(x)$, intercepts, & give an approximate sketch of the graph.

$y = (x+4)^5(x-3)^4$.



1. Y-intercept: Set $x=0$
   $\Rightarrow y = (4)^5(-3)^4 = 82,944$.
2. X-int. : Set $y=0$
   $(x+4)^5(x-3)^4 = 0$
   $\Rightarrow x=-4,\ x=3$.
3. $\lim_{x\to\infty} y = \lim_{x\to\infty} (x+4)^5(x-3)^4 = \infty$
4. $\lim_{x\to-\infty} y = \lim_{x\to-\infty} (x+4)^5(x-3)^4 = -\infty$

The End

EX_{11} $f(x) = (x-2)^4(x+1)^3(x-1)$

Sec. 2.7

Tangents, Velocities and other rates of change

Objectives
1. To define the slope of a tangent line
2. " " " Average velocity and instantaneous velocity
3. " " " average rate of change and " rate of change.

Def. The tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad \text{if exists}$$

Q-8, 156 Find the equ. of the tangent line to the curve at the point
$y = \sqrt{2x+1}$, $(4,3)$

$$m = \lim_{x \to 4} \frac{f(x) - f(4)}{x - 4} = \lim_{x \to 4} \frac{\sqrt{2x+1} - 3}{x - 4} \cdot \frac{\sqrt{2x+1} + 3}{\sqrt{2x+1} + 3}$$

$$= \lim_{x \to 4} \frac{2x+1-9}{(x-4)(\sqrt{2x+1}+3)} = \lim_{x \to 4} \frac{2(x-4)}{(x-4)(\sqrt{2x+1}+3)} = \frac{2}{3+3} = \frac{1}{3}$$

The tangent equ: $y - y_1 = m(x - x_1)$

$$y - 3 = \frac{1}{3}(x-4) \Rightarrow y = \frac{1}{3}x + \frac{5}{3}.$$

Note, Another expression for the slope at $P(a, f(a))$

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Q.14, 156 (a) Find m for $y = \frac{1}{\sqrt{x}}$ at the point where $x = a$

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{\frac{1}{\sqrt{a+h}} - \frac{1}{\sqrt{a}}}{h} = \lim_{h \to 0} \left[\frac{1}{h}\left(\frac{1}{\sqrt{a+h}} - \frac{1}{\sqrt{a}}\right)\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{\sqrt{a} - \sqrt{a+h}}{\sqrt{a}\sqrt{a+h}}\right] \cdot \frac{\sqrt{a} + \sqrt{a+h}}{\sqrt{a} + \sqrt{a+h}}$$

$$= \lim_{h \to 0} \frac{1}{\cancel{h}}\left[\frac{\cancel{a} - \cancel{a} - \cancel{h}^{\,-1}}{\sqrt{a}\sqrt{a+h}(\sqrt{a} + \sqrt{a+h})}\right] = \frac{-1}{\sqrt{a^2}(\sqrt{a} + \sqrt{a})} = \frac{-1}{2a^{3/2}} = -\frac{1}{2} a^{-3/2}$$

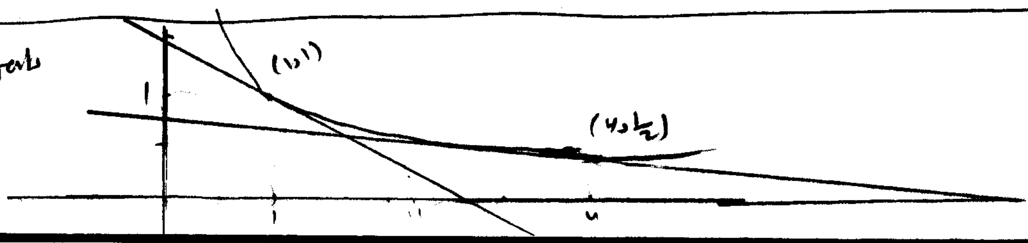
(b) Find the equ. at
(i) $(1,1)$

$$m = -\frac{1}{2}(1)^{-3/2} = -\frac{1}{2} \Rightarrow \text{The equ.: } y - 1 = -\frac{1}{2}(x-1)$$
$$y = -\frac{1}{2}x + \frac{3}{2}.$$

(ii) $(4, \frac{1}{2}) \Rightarrow m = -\frac{1}{2}(4)^{-3/2} = -\frac{1}{2}(2)^{-3} = \frac{-1}{2(2)^3} = \frac{-1}{16}$

$\therefore$ The equ.: $y - \frac{1}{2} = -\frac{1}{16}(x-4) \Rightarrow y = -\frac{1}{16}x + \frac{3}{4}$

(c) Graph both tangents



Def. The average velocity:

If an object moves along a straight line given by $S = f(t)$, when
S: Displacement from the origin at time t.
$f(t)$: position function

The average velocity in $[t=a, t=a+h]$ is:

$$\text{Av. Velocity} = \frac{\text{displacement}}{\text{time}} = \frac{f(a+h) - f(a)}{a+h-a} = \frac{f(a+h)-f(a)}{h}$$

Def. The velocity or instantaneous velocity is $v(a)$ at t is:

$$v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \equiv \text{The slope of the line at } t=a.$$

Q.20, 156 $S = t^2 - 8t + 18$, t in seconds, S in meters.

a) Find the av. veloc. over:

(i) $[3,4]$ Av. Velocity $= \dfrac{S(4) - S(3)}{4-3} = \dfrac{2-3}{1} = -1$ m/s.

(iii) $[4,5]$ Av. Velocity $= \dfrac{S(5) - S(4)}{5-4} = \dfrac{3-2}{1} = 1$ m/s.

b) Instantaneous velocity when $t=4$

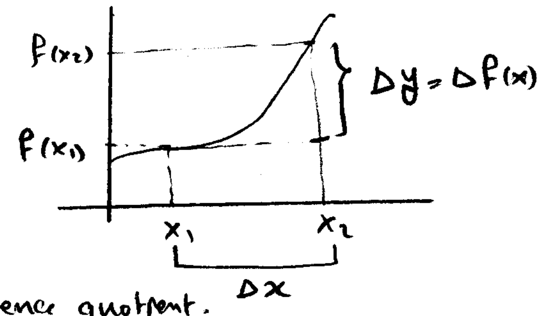
$$V(4) = \lim_{h\to 0} \frac{S(4+h) - S(4)}{h} = \lim_{h \to 0} \frac{(4+h)^2 - 8(4+h) + 18 - (2)}{h}$$

$$V(4) = \lim_{h\to 0} \frac{\cancel{16} + \cancel{8h} + h^2 - \cancel{32} - \cancel{8h} + \cancel{18} - \cancel{2}}{h} = \lim_{h\to 0} \frac{h^2}{h} = \lim_{h \to 0} h = 0$$

---

* Other Rates of Change. If $y = f(x)$ and x changes from x_1 to x_2
then the change in $x = \Delta x = x_2 - x_1$ (Increment)

and the corresponding change in y is
$\Delta y = f(x_2) - f(x_1)$



then the average rate of change of y with respect to x over $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \text{Difference quotient.}$$

And the instantaneous rate of change of y with respect to x at $x = x_1$

is $$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Q.24, 157

Year	1992	1994	1996	1998	2000
P	10,036	10,109	10,152	10,175	10,186

a. Av. rate of growth (i) From 1992 to 1996 $\Rightarrow \dfrac{P(1996) - P(1992)}{1996 - 1992} = \dfrac{10{,}152 - 10{,}036}{4} = \dfrac{116}{4} = 29$ thousands / Yrs.

(ii) 94–96 $\Rightarrow$ 21.5 thousd / Yrs

(iii) 96–98 $\Rightarrow$ 11.5 , / Yr

b. From (ii) & (iii) we have $= \dfrac{21.5 + 11.5}{2} = 16.5$ thous / Yrs

The End.

Derivatives

Objectives 1. To define the derivative at a number a
2. To interpret the derivative as tangent slope, velocity and rate of change.

Def. The derivative of a function f at a denoted by $f'(a)$ is:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad \text{if this limit exists.}$$

Another Formula: $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ (put $x = a+h$)

Q.18 / 163, Find $f'(a)$, $f(x) = \sqrt{3x+1}$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{\sqrt{3x+1} - \sqrt{3a+1}}{x-a} \cdot \frac{\sqrt{3x+1} + \sqrt{3a+1}}{\sqrt{3x+1} + \sqrt{3a+1}}$$

$$= \lim_{x \to a} \frac{3x + \cancel{1} - 3a - \cancel{1}}{(x-a)(\sqrt{3x+1} + \sqrt{3a+1})} = 3 \lim_{x \to a} \frac{(\cancel{x-a})}{(\cancel{x-a})(\sqrt{3x+1} + \sqrt{3a+1})} =$$

$$= \frac{3}{\sqrt{3a+1} + \sqrt{3a+1}} = \frac{3}{2\sqrt{3a+1}}$$

NOTE. The slope of the tangent line at $(a, f(a))$ is the derivative of f at a, i.e: $f'(a) = m$, and the equ.: $y - f(a) = f'(a)(x - a)$.

Q.8 / 163, If $g(x) = 1 - x^3$, find $g'(0) = ?$, an equ. of the tangent line at $(0,1)$

$$g'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{1 - (h+0)^3 - 1}{h} = \lim_{h \to 0} \frac{-h^3}{h}$$

$$= \lim_{h \to 0} -h^2 = 0$$

$\Rightarrow$ The tangent equ. is: $y - y_1 = m(x - x_1)$ but $m = g'(0) = 0$

$y - 1 = 0(x - 0) = 0 \Rightarrow y = 1$

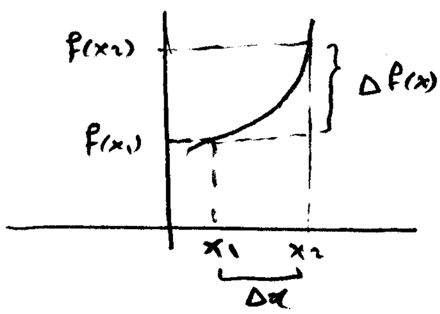
NOTE 1. The instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$ is the derivative of f at a.

$\Delta y = \Delta f(x) = f(x_2) - f(x_1)$

$\Delta x = x_2 - x_1$

Instantaneous rate of change $= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$

$$= \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$



→ NOTE

2. If $s = f(t)$ is the position function of a particle moves along a straight line, then the velocity or instantaneous velocity at $t = a$ is the derivative at a or $f'(a)$

3. The speed of the particle $= |f'(a)|$

Ex. 4, 161 The position of a particle is: $s = f(t) = \frac{1}{1+t}$, t in seconds, s in meters. Find the velocity and the speed after 2 seconds.

i) Velocity at $t=2 = f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{\frac{1}{3+h} - \frac{1}{3}}{h}$

$= \lim_{h\to 0} \frac{1}{h}\left[\frac{\cancel{3}-\cancel{3}-h}{3(3+h)}\right] = \lim_{h\to 0} \frac{-\cancel{h}}{\cancel{h}(3(3+h))} = \frac{-1}{3(3+0)} = -\frac{1}{9}$ m/s.

ii) The speed at $t=2 = |f'(2)| = \frac{1}{9}$ m/s.

Q. 27, 163: $C = f(x)$: Cost of producing x-ounces of gold from gold mine.

a) what is the meaning of $f'(x)$? The unit?

It means the rate of change of production cost with respect to the number of onces. The unit is dollars or Ryals per once.

b) $f'(800) = 17$: The cost of producing the first 800 onces of gold is 17 dollars.

c) $f'(x)$ will increase as x-onces of gold produced is increased.

EX: State f and a for the following limits.

Q. 19, 163: $\lim_{h\to 0} \frac{(1+h)^{10} - 1}{h}$ compare it with $\frac{f(a+h) - f(a)}{h}$ ⇒ $f(a+h) = (1+h)^{10}$ ⇒ $f(x) = x^{10}$

put x in place of 1+h or a+h $a = 1$.

Q. 23, 163: $\lim_{h\to 0} \frac{\cos(\pi+h) + 1}{h}$ compare it with $\frac{f(a+h) - f(a)}{h}$: $f(a+h) = \cos(\pi + h)$

⇒ $f(x) = \cos x$, $a = \pi$.

Q. 24, 163: $\lim_{t\to 1} \frac{t^2 + t - 2}{t-1}$ Compare it with $\lim_{t\to a} \frac{f(t) - f(a)}{t-a}$ ⇒ $a = 1$

$f(x) = x^2 + x$

The End

sec. 2.9

The Derivative as a Function

objectives:
1. To define the derivative at $(x, f(x))$
2. " " other symbols of derivative.
3. " " left-hand and right-hand derivatives.

Def: The derivative of $f(x)$ at x is given by:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

= The slope of the tangent at the point $(x, f(x))$.

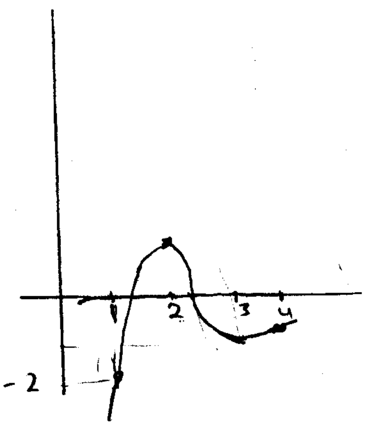
Q.1 173: Use the graph of f to estimate the values of each derivative. Sketch the graph of f'.

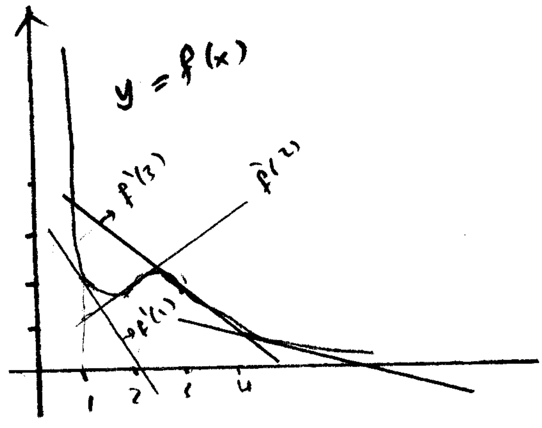
a) $f'(1) \approx -2$

b) $f'(2) \approx 0.8$

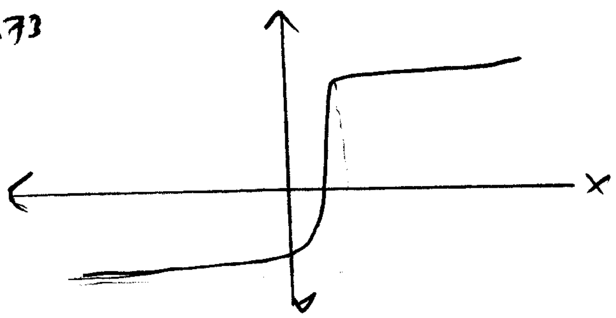
c) $f'(3) \approx -1$

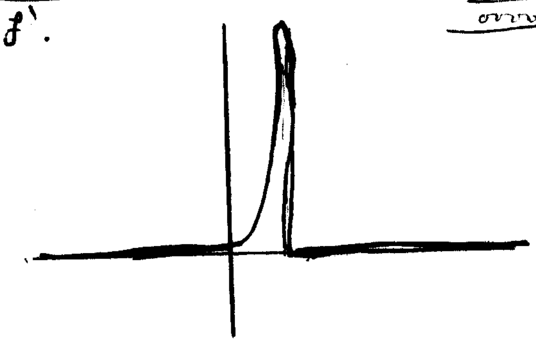
d) $f'(4) \approx -.5$

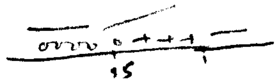




Q.8 173: use the graph of f to sketch f'.







Q.28 174: Find $f'(x)$ using the definition $f(x) = \frac{3+x}{1-3x}$ and state $\operatorname{dom}(f)$ & f'.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{3+x+h}{1-3x-3h} - \frac{3+x}{1-3x}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{(3+x+h)(1-3x) - (3+x)(1-3x-3h)}{(1-3x)(1-3x-3h)}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{\cancel{3}\cancel{-9x}+\cancel{x}\cancel{-3x^2}+h-\cancel{3xh}\cancel{-3}\cancel{+9x}+9h\cancel{-x}\cancel{+3x^2}\cancel{+3xh}}{(1-3x)(1-3x-3h)}\right]$$

$$= \lim_{h\to 0} \frac{1}{\cancel{h}}\left[\frac{10\cancel{h}}{(1-3x)(1-3x-3h)}\right] = \lim_{h\to 0} \frac{10}{(1-3x)(1-3x-3h)}$$

$$= \frac{10}{(1-3x)^2}.$$

$\operatorname{Dom}(f) = \mathbb{R} - \{\frac{1}{3}\} = \operatorname{Dom}(f')$

$= (-\infty, \frac{1}{3}) \cup (\frac{1}{3}, \infty)$

23, 174 $f(x) = 1 - 3x^2$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{1 - 3(x+h)^2 - (1-3x^2)}{h} =$$

$$= \lim_{h\to 0} \frac{\cancel{1} - \cancel{3x^2} - 6xh - 3h^2 - \cancel{1} + \cancel{3x^2}}{h} = \lim_{h\to 0} \frac{\cancel{h}(-6x - 3h)}{\cancel{h}} = -6x$$

$\text{Dom}(f) = (-\infty, +\infty)$, $\text{Dom}(f') = (-\infty, +\infty)$

Other Notations: If $y = f(x)$ then the derivative may be denoted by

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = D f(x) = D_x f(x).$$

D & $\frac{d}{dx}$ are called differentiation operators

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \quad \} \text{ Leibniz notation}$$

The derivative of f at $x=a$ is denoted by $\left.\frac{dy}{dx}\right|_{x=a}$ or $\left.\frac{dy}{dx}\right]_{x=a}$.

which is equivalent to $f'(a)$

Def: A function f is differentiable at a if $f'(a)$ exists. It is differentiable on an open interval (a,b) or $[a,\infty), (-\infty,a)$, or $(-\infty,+\infty))$ if it is differentiable at every number in the interval

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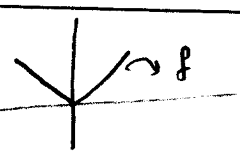
Ex.6 170: When is $f(x) = |x|$ differentiable?



$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$ there are three cases:

Case 1: If $x > 0$, $f(x) = x$. Choose h such that $x + h > 0$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{x+h-x}{h} = \lim_{h\to 0} \frac{h}{h} = \lim_{h\to 0} 1 = 1$$

$\therefore$ f is differentiable for any $x > 0$

Case 2: If $x < 0$; $f(x) = -x$, choose h so that $x+h < 0$

$$f'(x) = \lim_{h\to 0} \frac{-(x+h) - (-x)}{h} = \lim_{h\to 0} \frac{-x-h+x}{h} = \lim_{h\to 0} \frac{-h}{h} = -1.$$

$\therefore$ f is differentiable for any $x < 0$

Case 3: when $x = 0$

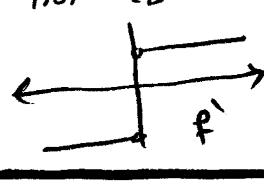
$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{|h| - 0}{h} = \lim_{h\to 0} \frac{|h|}{h}$$

But: $\lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = \lim_{h\to 0^+} 1 = 1$, $\lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0} \frac{-h}{h} = -1$.

$\Rightarrow \lim_{h\to 0} \frac{|h|}{h}$ does not exist $\Rightarrow f'(0)$ does not exist.

Thm4

$$\therefore f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \\ \text{d.N.E} & \text{if } x = 0 \end{cases}$$



Thm. If f is differentrable at a, then f is conts. at a

Note, The converse of theorem is not nessesrly true.

Ex. If $f'(3) = -3$, $f(3) = 5$, Find $\lim_{x \to 3} f(x)$

$f'(3)$ exist $\Rightarrow$ $f(x)$ is conts. at $x=3$

$\Rightarrow \lim_{x\to 3} f(x) = f(3) = 5$

Def, (Q.46 / 135). 1. The left-hand derotive of f at a is defind by

$$f'_-(a) = \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h}$$

2. The right-hand derivatin of f at a is defind by:

$$f'_+(a) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$$

If $f'_-(a)$, $f'_+(a)$ are exist and $f'_-(a) = f'_+(a)$ then $f'(a)$ exist.

Q.46 / 175, a) Find $f'_-(4)$, $f'_+(4)$ for $f(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 5-x & \text{if } 0 < x < 4 \\ \frac{1}{5-x} & \text{if } x \ge 4. \end{cases}$

?) $f'_-(4) = \lim_{h\to 0^-} \frac{f(4+h) - f(4)}{h}$, $h<0 \Rightarrow 4+h<4$

$= \lim_{h\to 0^-} \frac{5-(4+h) - (\frac{1}{5-4})}{h} = \lim_{h \to 0^-} \frac{1-h-1}{h}$

$= \lim_{h\to 0^-} \frac{-h}{h} = \lim_{h\to 0^-} -1 = -1$

$f'_+(4) = \lim_{h\to 0^+} \frac{f(4+h) - f(4)}{h}$, $h>0 \Rightarrow 4+h>4$

$= \lim_{h\to 0^+} \frac{\frac{1}{5-4-h} - \frac{1}{5-4}}{h} = \lim_{h\to 0^+} \frac{1}{h}\left[\frac{1}{1-h} - \frac{1}{1}\right]$

$= \lim_{h\to 0^+} \frac{1}{h}\left[\frac{1-1+h}{1(1-h)}\right] = \lim_{h\to 0^+} \frac{1}{1-h} = 1$

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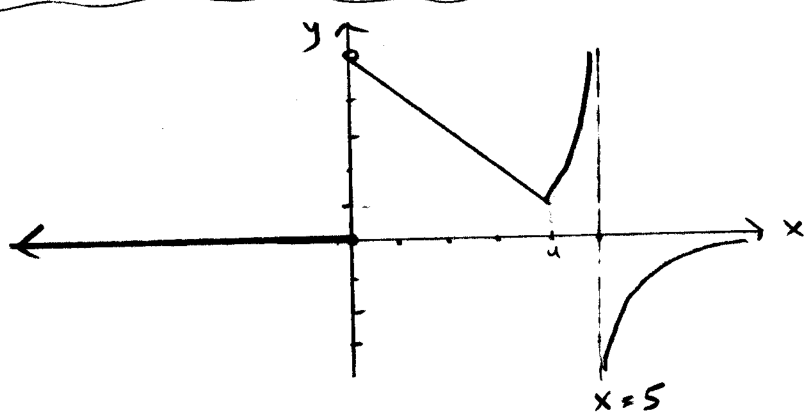
b.



→ Q.46 / 175, c. Where f is discont.

1. f is conts. on $(-\infty,0)$, $(0,4)$, $(4,5)$, $(5,\infty)$

2. At $x=0$: i) $f(0)=0$ ii) $\lim_{x\to 0^-} f(x) = 0 \neq \lim_{x\to 0^+} f(x) = 5 \Rightarrow \lim_{x\to 0} f(x)$ D.N.E

$\Rightarrow f$ is disconts. at $x=0$.

3. At $x=4$ i) $f(4)=1$ ii) $\lim_{x\to 4^-} f(x) = 1 = \lim_{x\to 4^+} f(x) = 1$

iii) $\lim_{x\to 4} f(x) = f(4)$

$\Rightarrow f$ is conts. at $x=4$.

4. At $x=5$ i) $f(5)$ undefined $\Rightarrow f$ is disconts. at $x=5$

d). where f is not diff..

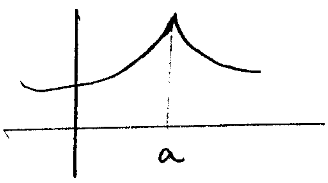
at $x=4$ becaus $f'_-(4) \neq f'_+(4)$

at $x=0,5$ becaus f is disconts.

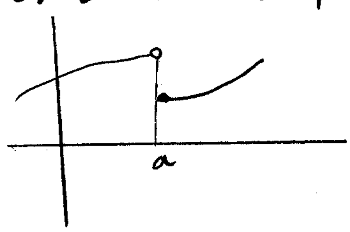
How can a func. fail to be differentiable?

$f(x)$ is not differentiable at $x=a$ when a is

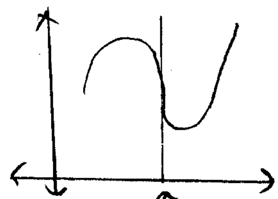
a) A corner point



b) Discontinuity point.

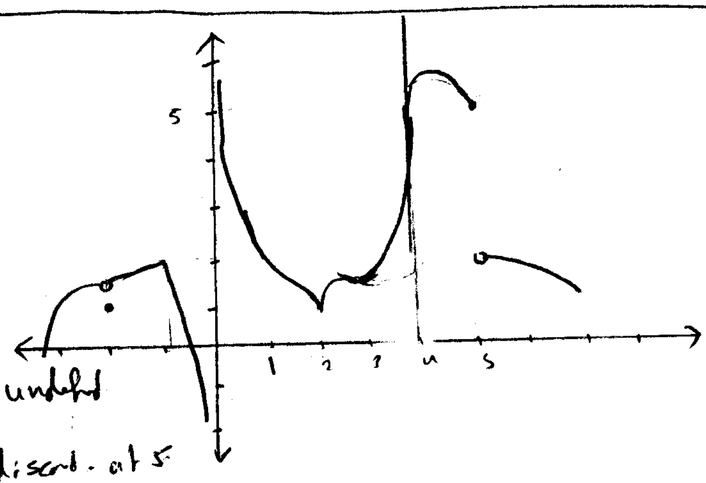


c) Vertical tangent



f is conts. at a and $\lim_{x\to a} |f'(x)| = \infty$

Q.38 / 175, Consider the graph of f.



a) At what points f is disc.? why

f is disconts. at $x=-2$, $x=0$, $x=5$

becaus 1) $\lim_{x\to -2} f(x) \neq f(-2)$ (removable disc.)

2) $\lim_{x\to 0} f(x)$ does not exist and $g(0)$ undefd

3) $\lim_{x\to 5} f(x)$ does " " and f is discont. at 5

b) At what points g is not diff.? why

f is not diff. at 1) $x=-2,0,5$ (not conts).

2) $x=-1$, (corner)

3) $x=2$ (vertical tangt)

4) $x=4$ (vertical tangt)

The End