Sec. 3.11

* Linear Approximations and Differentials

Objectives:
1. To define the linear approximation and solve problems
2. 〃 〃 〃 differentials 〃 〃 〃

Def: The linear approximation or tangent line approximation of f at a is:

$f(x) \approx f(a) + f'(a)(x-a)$.

The linear function is: $L(x) = f(a) + f'(a)(x-a)$

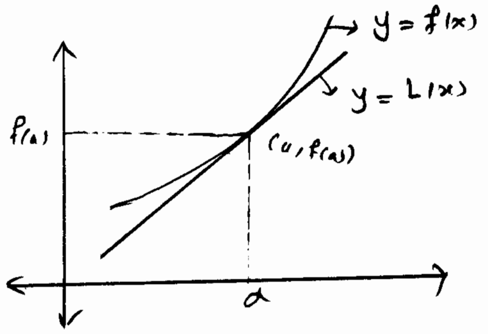
$y - y_1 = f'(x)(x - x_1)$

$y - f(a) = f'(a)(x-a)$

$y = f(a) + f'(a)(x-a)$

or $f(x) \approx f(a) + f'(a)(x-a) = L(x)$

$f(x) \approx L(x)$



Ex. Find the linearization L(x) of f at a.

Q.5, 267

$f(x) = x^3$, $a = 1$ $\qquad L(x) = f(x) + f'(a)(x-a)$

$f'(x) = 3x^2 \Rightarrow f'(1) = 3(1)^2 = 3$, $f(1) = (1)^3 = 1$

$\therefore L(x) = f(a) + f'(a)(x-a)$

$L(x) = 1 + 3(x-1) = 1 + 3x - 3 = 3x - 2$.

$\therefore f(x) \approx 3x - 2$.

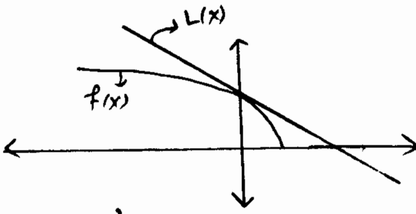
Q.9, 267 (i) Find the linear approximation: $f(x) = \sqrt{1-x}$ at $a = 0$.

$f'(x) = \frac{-1}{2\sqrt{1-x}} \Rightarrow f'(a) = f'(0) = \frac{-1}{2\sqrt{1-0}} = \frac{-1}{2}$

$f(a) = f(0) = \sqrt{1} = 1$

$\therefore L(x) = f(0) + f'(0)(x-0)$

$= 1 + (-\frac{1}{2})(x-0) = 1 - \frac{1}{2}x \Rightarrow \sqrt{1-x} \approx 1 - \frac{1}{2}x$.



(ii) Use it to approximate $\sqrt{.9}$ and $\sqrt{.99}$.

1) $\sqrt{.9} = \sqrt{1-.1} = \sqrt{.9} \approx 1 - \frac{1}{2}(.1) = 1 - .05 = 0.95$

2) $\sqrt{.99} = \sqrt{1-.01} = 1 - \frac{1}{2}(.01) = 1 - .005 = 0.995$.

Q.14, 267 verify the given linear approximation at a = 0. Determine the values of x such that it is accurate to within 0.1

$e^x = 1 + x$, $a = 0$

$f(x) = e^x \rightarrow f(0) = e^0 = 1$

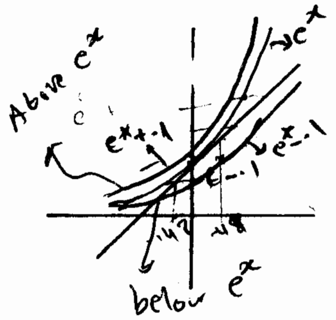
$f'(x) = e^x \Rightarrow f'(0) = e^0 = 1$

$\Rightarrow f(x) \approx f(a) + f'(a)(x-a)$

$= f(0) + f'(0)(x-0)$

$= 1 + 1(x-0) = 1 + x$.

$\therefore e^x \approx 1 + x$



NOTE. The accuracy is $|f(x) - L(x)| <$ error

$|e^x (1+x)| < .1 \Rightarrow -.1 < e^x - (1+x) < .1$

$-e^x - .1 < -(1+x) < .1 - e^x \Rightarrow e^x - .1 < 1 + x < e^x + .1$

$\Rightarrow -.483 < x < .416$ by the graph of the tangent.

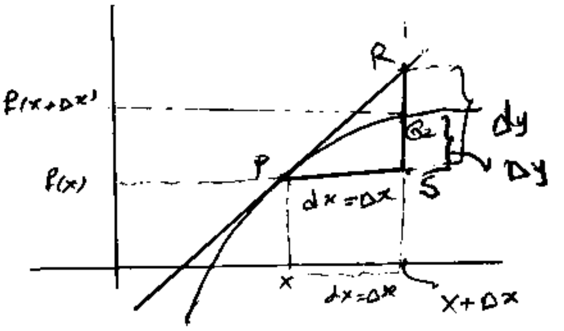
Sec 3.11

✱ Differentials

Def. The differential dy for the function $y = f(x)$ is given by

$$\frac{dy}{dx} = f'(x) \Rightarrow dy = f'(x)\,dx .$$

✱ Geometric Meaning:



Let $P\,(x, f(x))$ and $R\,(x+\Delta x, f(x+\Delta x))$ be two points of f.

Let $\Delta x = dx$. then,

$$\boxed{\Delta y = f(x+\Delta x) - f(x)}$$

The tangent slope $= f'(x)$: $SR = dy = f'(x)\,dx$.

The linear approximation of f at a can be written as:

$\boxed{f(a+dx) \approx f(a) + dy}$, $\boxed{\Delta y \approx dy = f'(x)\,dx}$.

Ex. a) Find dy b) Evaluate dy for the given x, dx.

Q.21 / 268: $y = x^2+2x$, $x=3$, $dx=\frac{1}{2}$.

(a) $dy = f'(x)\,dx = (2x+2)\,dx$

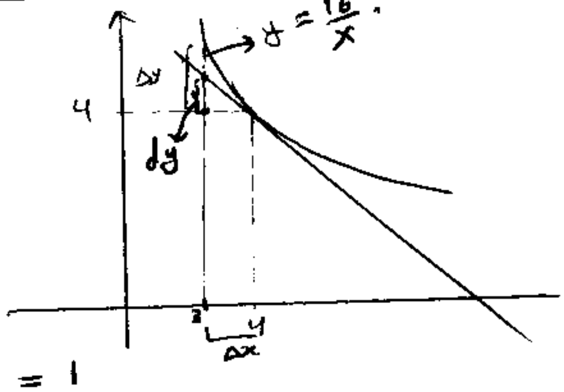
(b) when $x=3$, $dx=\frac{1}{2} \Rightarrow dy = (2(3)+2)(\frac{1}{2}) = 4$

Q.22 / 268: $y = e^{x/4}$, $x=0$, $dx = 0.1$

(a) $dy = \frac{1}{4} e^{x/4}\,dx$

(b) $dy = \frac{1}{4} e^{0} (.1) = \frac{1}{4}(1)(.1) = \frac{.1}{4} = 0.025$

Ex. Compare Δy, dy. when $dx = \Delta x$.



Q.30 / 268: $y = \frac{16}{x}$, $x=4$, $\Delta x = -1$

a) $\Delta y = f(x+\Delta x) - f(x) = f(4-1) - f(4)$
$= f(3) - f(4) = \frac{16}{3} - \frac{16}{4} = \frac{4}{3}$.

b) $dy = \frac{-16}{x^2}\,dx$

when $x=4$, $dx = \Delta x = -1 \Rightarrow dy = \frac{-16}{(4)^2}\cdot(-1) = 1$

Ex: Use differentials (or linear approximation) to estimate,

Q.32 / 268: $\sqrt{99.8}$ Let $y = \sqrt{x} \Rightarrow dy = \frac{1}{2\sqrt{x}}\,dx$

when $x = 100$, $dx = -0.2 \Rightarrow dy = \frac{1}{2\sqrt{100}}(-.2) = \frac{-.1}{10} = -.01$

$f(a+dx) \approx f(a)+dy$
$= f(100) + dy$
$= 10 + (-.01) = 9.99$

Sec. 3.11

Ex.5 266: $r = 21$ cm with a possible error of at most 0.05 cm

What is the maximum error in using r to complete the volume of the sphere

$V = \frac{4}{3}\pi r^3$ $r = 21$, error $= dr = \Delta r = .05$, then

$$dv = 4\pi r^2 \, dr$$

$$dv = 4\pi(21)^2(.05) \approx 277$$

∴ The maximum error in the calculated volume is about 277 cm^3.

Def: The relative error is of $y = f(x)$ is: $\frac{\Delta y}{y} \approx \frac{dy}{y}$

EX: for **Ex.5**: The relative error is:

$$\underbrace{\frac{\Delta V}{V} \approx \frac{dv}{V}}_{\text{Relative error in the volume}} = \frac{4\pi r^2 \, dr}{\frac{4}{3}\pi r^3} = \boxed{3}\underbrace{\frac{dr}{r}}_{\text{Relative error in the radius.}}$$

Relative error in radius $= \frac{dr}{r} = \frac{0.05}{21} \approx 0.0024$

Def. The percentage error = relative error * 100%.

For **Ex.5** percentage error = 0.0024 * 100% = 0.24%

Relative error in volume $= \frac{dv}{V} = 3\frac{dr}{r} = 3(.0024) = 0.007$

The percentage error in volume = 0.007 * 100% = 0.7%

Q-41 268: edge = 30 cm with an error of 0.1 cm. use differentials to estimate:

i) maximum possible error ii) Relative error iii) percentage error in

a) The volume of the cube: let x be the length of the cube edge.

i) $V = x^3 \Rightarrow dv = 3x^2 \, dx$, $dx = \Delta x = .1$, $x = 30$

∴ Max. error in volume $= dV = 3(30)^2(.1) = 270$ cm^3.

ii) Relative error $= \frac{dV}{V} = \frac{3x^2 dx}{x^3} = 3\frac{dx}{x} = \frac{3(.1)}{30} = 0.01$

iii) percentage error in volume = 0.01 * 100% = 1%.

b) The surface area of the cube: let S be the surface area.

i) $S = 6x^2 \Rightarrow ds = 12x \, dx$, $x = 30$, $dx = .1$

∴ Max. error in S $= ds = 12(30)(.1) = 36$ cm^2

ii) Relative error in S $\approx \frac{ds}{S} = \frac{12x\,dx}{6x^2} = 2\frac{dx}{x} = \frac{2(.1)}{30} = 0.00\bar{6}$

iii) Percentage error in S $= 0.00\bar{6}$ * 100% $= 0.\bar{6}$%.

The End

CH.4

* Applications Of Differentiation *

Sec. 4.1

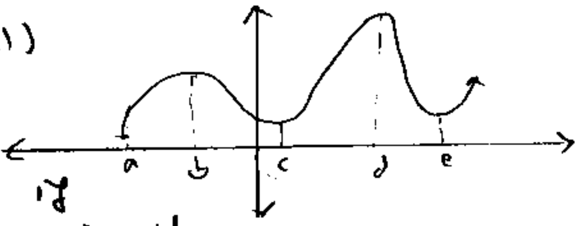
* Maximum and Minimum Values *

objectives 1. To define the absolute and local maximum or minimum values

2. " " " critical number(s) of a function f

3. " Consider the closed interval method to find the absolute Max. or min. of f.

Def. If f is a function with domain D, and c is a number in D, then:



1. f(x) has an absolute maximum (or global) at c if $f(c) \geqslant f(x)$ for all x in D where f(c) is the max. value of f.
2. f(x) has an absolute minimum at c if $f(c) \leq f(x)$ for all x in D and f(c) is the minimum value of f

NOTE, The maximum and minimum values of f are called extreme values of f.

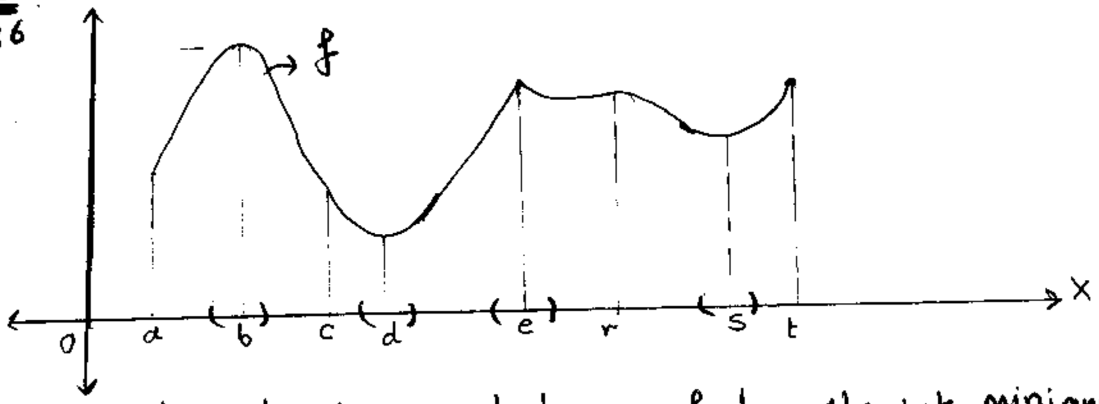
For the graph: 1. (a, f(a)) is the lowest point, so f has an absolute min. at x=a with value f(a)

2. (d, f(d)) is the highest point of f, so f has an abolute Max. at x =a with value f(a)

Def. 1. A function f has a local maximum (or relative maximum) at c if $f(c) \geqslant f(x)$ for all x in some open interval Containing c.

2. f has a local minimum at c if $f(c) \leq f(x)$ for all x in some open interval Containing c.

Q-3 Find extreme values and local values.
286



1. f has absolute max. at b 2. f has absolute minimum at d
3. f " local max. at b, e 4. " " local minimum at d and s
5. f has neither max. nor min. at a, c, r and t.

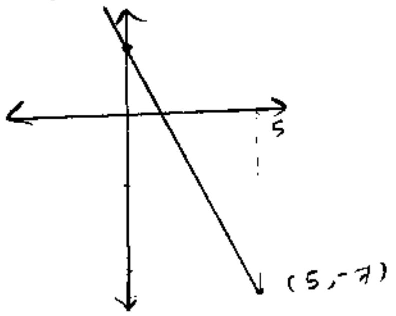
Sec. 4.1

NOTE: Every extreme is a local, but the opposite is not true. provided that it doesnt occore at an endpoint.

Ex: Sketch the graph of f and use it to find the absolute and local values.

Q.16 / 286: $f(x) = 3-2x \ , \ x \le 5$

x	5	0
y	-7	3

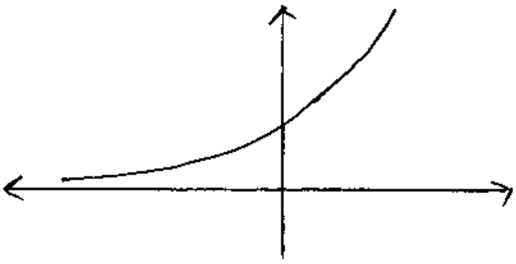


f has absolute minimum at $x=5$

But there is no local min. or absolute and local maximum.

Q.28 / 286: $f(x) = e^x$

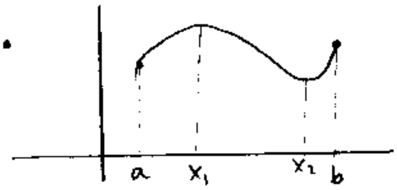
f(x) has no absolute or local maximum or minimum



The Extreme Value Theorem: If f is continuous on closed interval $[a,b]$, then f attains an absolute max. value $f(c)$ and an absolute min. value $f(d)$ at some numbers c,d in $[a,b]$

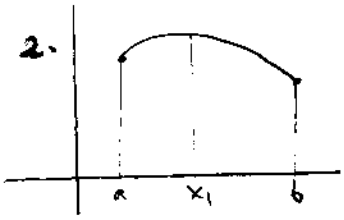
Ex: Consider the following graphs:

1.



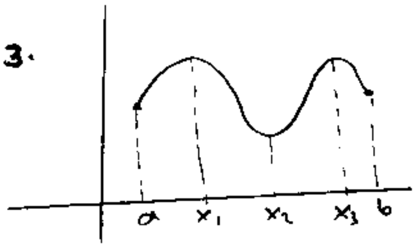
$f(x_1)$ is absolute max.
$f(x_2)$ = " min.

2.



$f(x_1)$ is absolute max.
$f(b)$ = " min.

3.



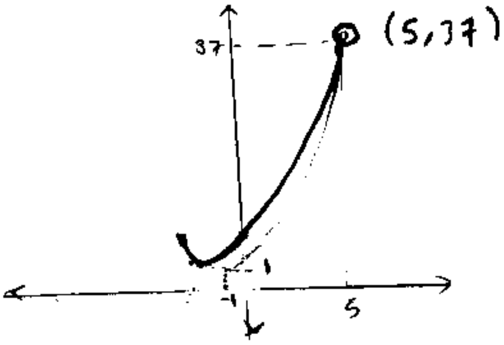
$f(x_1), f(x_3)$ are absolute max. $f(x_2)$ is " min.

Note Extreme values can be taken on more than once.

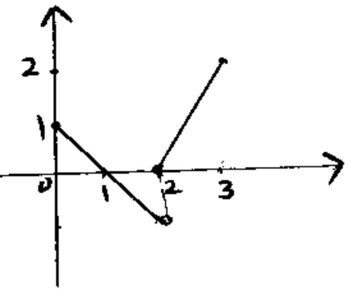
Ex: sketch and find extreme and local values.

Q.22 / 286: $f(x) = 1+(x+1)^2 \ , \ -2 \le x < 5$

1. $f(x)$ has absolute minimum when $x=-1$ with value $f(-1)=1$
2. f has local min = 1
3. f has no absolute or local maximum.



Q.29 / 286, $f(x) = \begin{cases} 1-x, & \text{if } 0 \le x < 2 \\ 2x-4, & \text{if } 2 \le x \le 3 \end{cases}$



Dom$(f) = [0,3]$ f is not cnts. at $x = 2$ But it has absolute max. at $x = 3$.

For $0 \le x < 2$:

x	0	2
y	1	-1

For $2 \le x \le 3$:

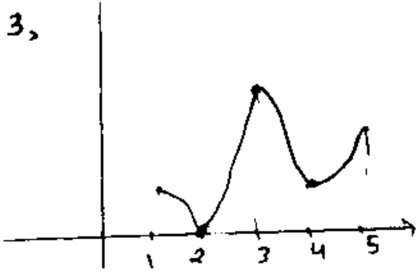
x	2	3
y	0	2

1. $f(x)$ has absolute max. at $x = 3$ with value $f(3) = 2$., but no local max.
2. " " no local or absolute min.

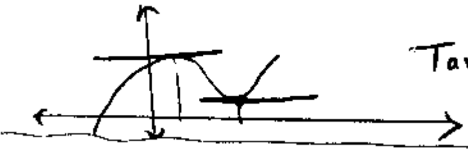
Ex. Sketch the graph of f that is cnts. on [1,5] and has:

Q.7 / 286, Absolute min. at 2, absolute max. at 3, local min. at 4

The graph may be as ⇒ .



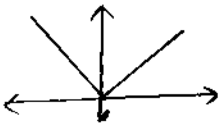
Fermat's Theorem: If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$



Tangents are horizontal

Note on Fermat's theorem: 1. If $f'(c)$ exists, then it 0, but $f(x)$ may have local or absolute max or min. but $f'(c)$ does not exist.

Ex.6 / 283 Consider $f(x) = |x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$



$f(x)$ has absolute min. at $x = 0$ with value $f(0) = 0$

but $f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$ and $f'(0)$ does not exist.

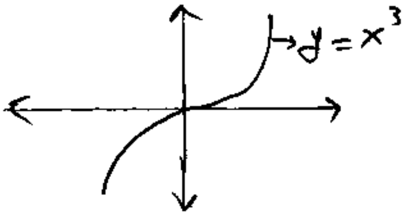
2- $f'(c)$ may be (0), but f has no local max or min. at c

Ex.5 / 282, Consider $f(x) = x^3$

$f'(x) = 3x^2$

$f'(0) = 0$

but f has no local max. or min. at $x = 0$



Def. If c is a number in domain f, then c is a critical number of f if either $f'(c) = 0$ or $f'(c)$ does not exist.

Ex. Find the critical numbers

Q.32 / 287, $f(x) = x^3 + x^2 - x \Rightarrow f'(x) = 3x^2 + 2x - 1$

Set $f'(x) = 0 \Rightarrow 3x^2 + 2x - 1 = 0 \Rightarrow (x+1)(3x-1) = 0 \Rightarrow x = -1, x = \frac{1}{3}$

are the only critical numbers

Q.39 287 , $g(t) = 5t^{2/3} + t^{5/3}$

$g'(t) = \frac{10}{3} t^{-1/3} + \frac{5}{3} t^{2/3}$

Critical number when → $g'(t) = 0 \Rightarrow \frac{10}{3} t^{-1/3} + \frac{5t^{2/3}}{3} = 0 \quad (t \neq 0)$

$\frac{5}{3} t^{-1/3}(2+t) = 0$

$\frac{5t^{-1/3}}{3} \neq 0 \Rightarrow \frac{5}{3t^{1/3}} \neq 0$

or $2 + t = 0 \Rightarrow t = -2$.

→ or $g'(t)$ does not exist $\Rightarrow g'(0)$ d.N.E.

∴ The critical numbers are $-2, 0$.

---

c. $g(x) = x - \sqrt{x}$

$g'(x) = 1 - \frac{1}{2\sqrt{x}} \Rightarrow$ set $g'(x) = 0 \Rightarrow 1 = \frac{1}{2\sqrt{x}} \Rightarrow \sqrt{x} = \frac{1}{2} \Rightarrow x = \frac{1}{4}$.

$g'(x)$ d.N.E. $\Rightarrow \sqrt{x} = 0 \Rightarrow x = 0$.

⇒ Critical numbers are $\{0, \frac{1}{4}\}$.

Fact. If f has a local maximum or minimum at c, then c is a critical number.

Becaus at local max. or min. the derivative is zero.

Absolute max or min of f over $[a,b]$: Closed Interval Method.

If f is conts. on $[a,b]$, then to find the absolute max. and min :

1. Find the value of f at all critical numbers on (a,b)
2. 〃 〃 〃 〃 〃 〃 the endpoints
3. The largest value is the absolute max. and the smallest is the absolute min.

Ex. Find the absolute max. and min. of f

Q.49 287 , $f(x) = x^3 - 3x + 1$, $[0,3]$

$f'(x) = 3x^2 - 3$

Set $f'(x) = 0 \Rightarrow x^2 = 1 \Rightarrow x = 1$ or $x = -1 \notin [0,3)$

1 is The critical number.

$f(1) = \underline{1 - 3 + 1 = -2}$, $f(0) = 1$, $f(3) = 27 - 9 + 1 = 19$

∴ 19 is the absolute max. of f.

-2 〃 〃 〃 min. 〃 f.

Q.54 287 , $f(x) = \frac{x^2 - 4}{x^2 + 4}$, $[-4, 4]$

$f'(x) = \frac{(x^2+4)(2x) - (x^2-4)(2x)}{(x^2+4)^2} = \frac{\cancel{2x^3} + 8x - \cancel{2x^3} + 8x}{(x^2+4)^2} = \frac{16x}{(x^2+4)^2}$

Set $f'(x) = 0 \Rightarrow 16x = 0 \Rightarrow x = 0$

∴ $f(0) = -1$, $f(-4) = f(4) = \frac{3}{5}$

∴ $\frac{3}{5}$ Abs. max. , -1 is Abs. min.

Q.57 / 287, $f(x) = \sin x + \cos x \ , \ [0, \frac{\pi}{3}]$

$f'(x) = \cos x - \sin x$

Set $f'(x) = 0 \Rightarrow \cos x = \sin x \Rightarrow \tan x = 1 \ (\cos x \neq 0)$

$\Rightarrow x = \frac{\pi}{4}$.

$f(0) = 1, \quad f(\frac{\pi}{4}) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \frac{2\sqrt{2}}{2} = \underset{\approx 1.41}{\sqrt{2}}, \quad f(\frac{\pi}{3}) = \frac{\sqrt{3}}{2} + \frac{1}{2} = \underset{\approx 1.37}{\frac{\sqrt{3}+1}{2}}$

$\therefore \sqrt{2}$ is the absolute max. and 1 is the absolute min.

Q.60 / 287, $f(x) = \frac{\ln x}{x} \ , \ [1, 3]$

$f'(x) = \frac{x \cdot \frac{1}{x} - \ln x \cdot 1}{(x)^2} = \frac{1 - \ln x}{x^2}$.

Set $f'(x) = 0 \Rightarrow 1 - \ln x = 0 \Rightarrow \ln x = 1 \Rightarrow x = e^1 = e$.

$f(1) = 0, \quad f(e) = \frac{\ln e}{e} = \frac{1}{e} = \underset{\approx .368}{e^{-1}}, \quad f(3) = \frac{\ln 3}{3} \approx 0.366$

$\therefore e^{-1}$ is the absolute max. and 0 is the absolute min.

The End.

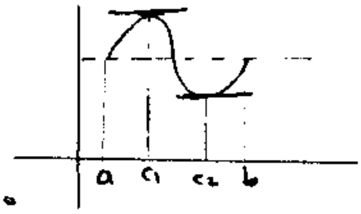
Sec. 4.2

* The Mean Value Theorem *

objectives

1. To introduce Rolles Theorem
2. " " the mean value theorem
3. " Solve some problems on the two theorems

Rolles Theorem, Let f be a function satisfies the following:



1. f is Conts. on $[a,b]$
2. " " differentiable on (a,b)
3. $f(a) = f(b)$

Then there is a number $c \in (a,b)$ such that $f'(c) = 0$

Note. If f satisfies Rolles theorem, then it means

1. There is a root for $f'(x)$ over (a,b).
2. " " a horizontal tangent of f over (a,b). (geometric meaning)

Q.3 / 295, $f(x) = \sin 2\pi x$, $[-1,1]$. verify and find c

1. $f(x)$ is Conts. on $[-1,1]$
2. $f'(x) = 2\pi \cos 2\pi x \Rightarrow$ diff. over $(-1,1)$
3. $f(-1) = 0$, $f(1) = 0 \Rightarrow f(-1) = f(1)$

$\therefore$ f satisfies Rolles theorem

$\Rightarrow$ There is at least $c \in (-1,1)$ such that $f'(c) = 0$

$$2\pi \cos 2\pi c = 0 \Leftrightarrow \cos 2\pi c = 0$$

$$\Rightarrow 2\pi c = \frac{\pi}{2} + 2k\pi \quad \text{or} \quad \Rightarrow c = \frac{\pi}{2(2\pi)} + \frac{2k\pi}{2\pi} = \frac{1}{4} + k \text{ , k integer}$$

$$\text{or} \quad 2\pi c = -\frac{\pi}{2} + 2k\pi \Rightarrow c = -\frac{1}{4} + k \text{ , k integer .}$$

$k = 0 \Rightarrow c = \frac{1}{4}$ or $c = -\frac{1}{4}$.

$k = 1 \Rightarrow c = \frac{5}{4} \notin (-1,1)$ or $c = -\frac{1}{4} + 1 = \frac{3}{4}$.

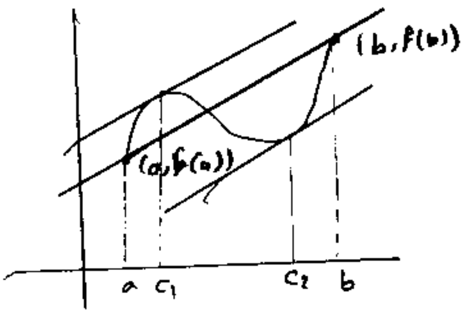
$k = -1 \Rightarrow c = -\frac{3}{4}$ or $c = -\frac{5}{4} \notin (-1,1) \Rightarrow$ c values are: $-\frac{1}{4}, -\frac{3}{4}, \frac{1}{4}, \frac{3}{4}$.

The Mean Value Theorem: Let f be a function satisfies the following:

1. f is Conts. on $[a,b]$
2. f " differentiable over (a,b) , then

there is at least c such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



Geometric Meaning. There is a tangent parallel to the line passes $(a, f(a))$ and $(b, f(b))$

NOTE. Rolles Theorem is a special case of the mean value theorem. When $f(a) = f(b)$.

Sec 4.2

Q.13 / 295, Verify M.V.T and find c. $f(x) = e^{-2x}$, $[0,3]$

1. $f(x)$ is conts. on $[0,3]$
2. $f'(x) = -2e^{-2x}$ ⇒ It is diff. over $(0,3)$.

∴ $f(x)$ satisfies the M.V.T then there is exist $c \in (0,3)$ such that $f'(c) = \frac{f(3)-f(0)}{3-0} \Rightarrow -2e^{-2c} = \frac{e^{-6}-1}{3}$ $(\div -2)$

$$\Rightarrow e^{-2c} = \frac{1-e^{-6}}{6} \Rightarrow -2c = \ln\left(\frac{1-e^{-6}}{6}\right)$$

$$\Rightarrow c = -\frac{1}{2}\ln\left(\frac{1-e^{-6}}{6}\right) \approx 0.897 \in (0,3)$$

Ex.5 / 293, Suppose that $f(0) = -3$ and $f'(x) \leq 5$ for all x. How large can $f(2)$ possibly be?

The func. is $f(x)$, $x \in [0,2]$

Because $f'(x)$ exists ⇒ it is conts.

∴ There exist a number $c \in (0,2)$ such that

$$f'(c) = \frac{f(2)-f(0)}{2-0}$$

$$\Rightarrow f(2) - f(0) = 2f'(c) \Rightarrow f(2) = f(0) + 2f'(c) = -3 + 2f'(c)$$

But $f'(c) \leq 5$

$2f'(c) \leq 10$

$-3 + 2f'(c) \leq -3+10 = 7 \rightarrow f(2) \leq 7$

∴ The large possible value of $f(2)$ is 7.

Q.17 / 295, Show that the equation $1 + 2x + x^3 + 4x^5 = 0$ has exactly one real root.

let $f(x) = 4x^5 + x^3 + 2x + 1$, $f(0) = 1$, $f(-1) = -4-1-2+1 = -6 < 0$

let $[a,b] = [-1,0]$

1. f is conts. over $[-1,0]$
2. $f(-1) = -6 < 0 < f(0) = 1$ ∴ f satisfies the I.V.T. $N = 0$.

∴ There is at least one root for $f(x)$ on $(-1,0)$

showing f has only one root:

Suppose that there are two roots a and b ($f(a) = f(b) = 0$) where $a<b$

Then f satisfies the conditions of Rolle's theorem

⇒ There exist $r \in (a,b)$ such that $f'(r) = 0$

But $f'(r) = 20r^4 + 3r^2 + 2 \geqslant 0$ for all $r \in (-1,0)$

⇒ There is only one root.

Q.25 / 295, Does there exist a function f such that $f(0) = -1$, $f(2) = 4$, $f'(x) \leq 2$ for all x?

Suppose that f exist., f is conts. on $[0,2]$ and diff. on $(0,2)$

∴ f satisfies the M.V.T ⇒ $\exists\, c \in (0,2)$ such that

$f'(c) = \frac{f(2)-f(0)}{2-0} = \frac{4-(-1)}{2} = \frac{5}{2} = 2.5$ but $f'(x) \leq 2 \Rightarrow f'(c) \leq 2$

⇒ There is no such function exists.

Theorem If $f'(x)=0$ for all x in (a,b) then f is constant on (a,b)

Proof let $x_1 < x_2$ be two points on $[a,b]$

1. f is conts. on $[x_1,x_2]$
2. " " diff. on (x_1,x_2) f satisfies the M.V.T

$\Rightarrow$ Then is at least $c \in (x_1,x_2)$ such that

$$f'(c) = \frac{f(x_2)-f(x_1)}{x_2-x_1} = 0$$

$$\Rightarrow f(x_2)-f(x_1)=0 \Rightarrow f(x_2)=f(x_1)$$

$$\Rightarrow f \text{ is constant.}$$

Corollary If $f'(x) = g'(x)$ for all x in (a,b) then $f-g$ is a constant on (a,b) or $f(x) = g(x)+C$ (or $f(x)-g(x)=C$.

Ex. 6 / **294**, Prove the identity: $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$.

Let $f(x) = \tan^{-1}x + \cot^{-1}x$ conts.

$$f'(x) = \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0 \Rightarrow f(x)=c.$$

$f(x)=c \Rightarrow$ let $x=1 \Rightarrow \tan^{-1}1 + \cot^{-1}1 = c \Rightarrow \frac{\pi}{4}+\frac{\pi}{4} = c \Rightarrow c = \frac{\pi}{2}$.

$$\therefore \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$$

Q. 32 / **296**, Prove the identy: $2\sin^{-1}x = \cos^{-1}(1-2x^2)$, $x \geq 0$

Let $f(x) = 2\sin^{-1}x - \cos^{-1}(1-2x^2)$

$$f'(x) = \frac{2}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-(1-2x^2)^2}} \cdot 4x$$

$$= \frac{2}{\sqrt{1-x^2}} - \frac{4x}{\sqrt{4x^2(1-x^2)}}$$

$$= \frac{2}{\sqrt{1-x^2}} - \frac{\overset{2}{\cancel{4x}}}{{}_{1}\cancel{2x}\sqrt{1-x^2}} = 0$$

$(1-2x^2)^2 = 1 - 4x^2 + 4x^4$

$1-(1-2x^2)^2 = 1-1+4x^2+4x^4$

$= 4x^2 - 4x^4$

$= 4x^2(1-x^2)$

$$\Rightarrow \underset{f}{2\sin^{-1}x} - \underset{g}{\cos^{-1}(1-2x^2)} = \text{Constant} = C$$

$$= C$$

put $x=1 \Rightarrow 2\sin^{-1}1 - \cos^{-1}(-1) = c$

$$2 \cdot \frac{\pi}{2} - (\pi) = c \Rightarrow \pi - \pi = c \Rightarrow c = 0$$

$$\therefore 2\sin^{-1}x - \cos^{-1}(1-2x^2) = 0 \Rightarrow 2\sin^{-1}x = \cos^{-1}(1-2x^2).$$

Ex, why $f(x) = |x+1|$ does not have $c \in (-2,0)$ Such that $f'(c)=0$.?

1. f is conts on $[-2,0]$ 2. $f(-2) = f(0) = 1$

3. $f'(x) = \begin{cases} 1 & \text{if } x > -1 \\ -1 & \text{if } x < -1 \\ \text{d.N.E} & \text{if } x = -1 \end{cases}$ So f is not diff. on $(-2,0)$

Rolles theorem

Sec. 4.3

How Derivatives Affect the Shape of a Graph

Objectives, 1. To find the intervals for which a function is increasing or decreasing.

2. To use the first derivative test to find the local max. and min. values

3. To find the intervals for which a function is concave upward or downward, and define the inflection point.

4. To use the second derivative test to find local max. or min.

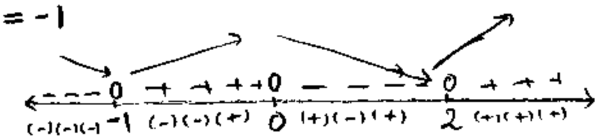
Def. Increasing/Decreasing Test. f is increasing if $x_1 < x_2$ then $f(x_1) < f(x_2)$
〃 〃 decreasing if $x_1 < x_2$ 〃 $f(x_1) > f(x_2)$

a) if $f'(x) > 0$ on an interval, then f is increasing on that interval.

b) 〃 $f'(x) < 0$ 〃 〃 〃, 〃 〃 〃 decreasing 〃 〃 〃.

Ex.1 297, where $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ increasing and decreasing.

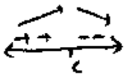
$$f'(x) = 12x^3 - 12x^2 - 24x \overset{set}{=} 0 \quad (\div 12)$$
$$x^3 - x^2 - 2x = 0$$
$$x(x^2 - x - 2) = 0 \Rightarrow x(x-2)(x+1) = 0$$
$$\Rightarrow x = 0,\ x = 2,\ x = -1$$

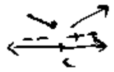


$f'(x) < 0$ for all $x < -1$ or $0 < x < 2$
∴ $f(x)$ is decreasing on $(-\infty, -1), (0, 2)$

$f'(x) > 0$ for all $-1 < x < 0$ or $x > 2$
∴ $f(x)$ is increasing on $(-1, 0), (2, \infty)$.

The First Derivative Test, Suppose that c is a critical number of a conts. function f, then:

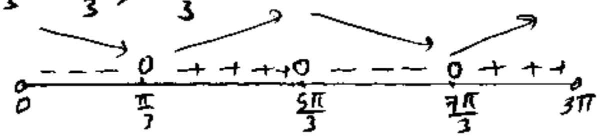
a. If f' changes from +ve. to -ve. at c, then f has local max. at c


b. 〃 〃 〃 〃 -ve. 〃 +ve. 〃 〃, 〃 〃 〃 〃 min. 〃 c


c. 〃 〃 does not change sign at c, then f has no local max. or min. at c.

Q.15 305, $f(x) = x - 2\sin x$, $0 < x < 3\pi$. Find the intervals of incr. or dec. and local valu

$f'(x) = 1 - 2\cos x \overset{set}{=} 0 \Rightarrow \cos x = \frac{1}{2} \rightarrow x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$.



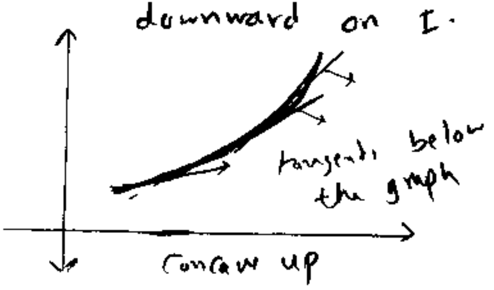
a) $f(x)$ is increasing on $(\frac{\pi}{3}, \frac{5\pi}{3}), (\frac{7\pi}{3}, 3\pi)$
$f(x)$ 〃 decreasing on $(0, \frac{\pi}{3}), (\frac{5\pi}{3}, \frac{7\pi}{3})$

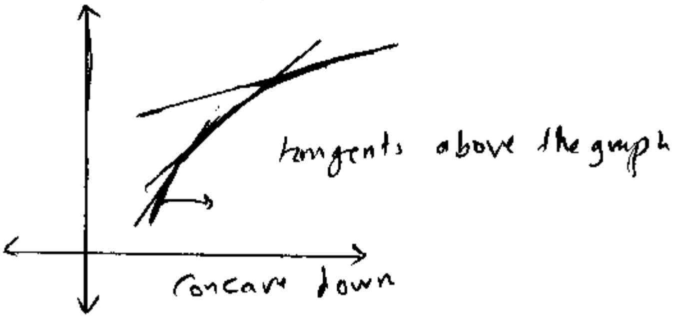
b) f has a local min. at $x = \frac{\pi}{3}, \frac{7\pi}{3}$ with values $f(\frac{\pi}{3}) = \frac{\pi}{3} - \sqrt{3}$, $f(\frac{7\pi}{3}) = \frac{7\pi}{3} - \sqrt{3}$
〃 〃 〃 max. at $x = \frac{5\pi}{3}$ with value $f(\frac{5\pi}{3}) = \frac{5\pi}{3} + \sqrt{3}$.

Sec. 4.3

Def.

1. If the graph of f lies above all of its tangents on an interval I, then it is called Concave upward on I.
2. If the graph of f lies below all its tangent on I, it is called Concave downward on I.





Concavity Test:

a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I

b) 〃 $f''(x) < 0$ 〃 〃 〃 〃 〃 〃 〃 〃 〃 〃 〃 〃 downward on I

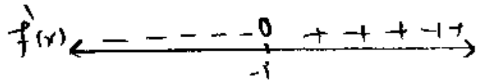
Def. A point P on the curve $y = f(x)$ is called an inflection point if f is conts. and changes from concave upward to downward or from downward to upward.

Ex. a) Find the intervals of decr. or incr.
b) 〃 〃 local max. and min.
c) 〃 〃 intervals of concavity and inflection points

Q.17, 305 $f(x) = xe^x$.

a) $f'(x) = xe^x + e^x \overset{set}{=} 0 \Rightarrow e^x(x+1) = 0 \Rightarrow e^x \neq 0,\ x+1 = 0 \Rightarrow x = -1$

$f(x)$ is increasing on $(-1, \infty)$
〃 〃 decreasing on $(-\infty, -1)$

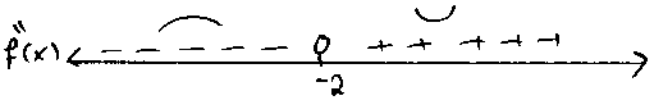


b) $f(x)$ has a local min. at $x = 1$ with value $f(-1) = -1e^{-1} = -e^{-1}$.

c) $f''(x) = e^x(1) + (x+1)e^x \overset{set}{=} 0 \Rightarrow e^x(1 + x + 1) = 0$

$e^x(2 + x) = 0 \Rightarrow e^x \neq 0,\ x + 2 = 0 \Rightarrow x = -2$

$f(x)$ is concave upward on $(-2, \infty)$
〃 〃 〃 downward 〃 $(-\infty, -2)$



$f(x)$ has an inflection point at $(-2, f(-2)) = (-2, -2e^{-2})$

The second derivative Test: Suppose f'' is continuous near c.

a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local min. at c
b) 〃 $f'(c) = 0$ 〃 $f''(c) < 0$, 〃 〃 〃 〃 max. 〃 c

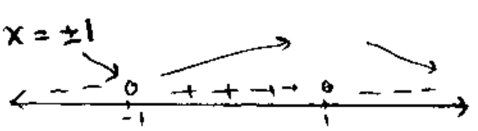
Note, If $f'(c) = 0$, $f''(c) = 0$, the test is inconclusive

Sec. 4.3

Q.34, 305 $f(x) = 2 + 3x - x^3$

a) I/D intervals: $f'(x) = 3 - 3x^2 \overset{set}{=} 0$

$\Rightarrow 3 = 3x^2 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$



f(x) is increasing on (-1,1) and decreasing on $(-\infty,-1)$, $(1,\infty)$

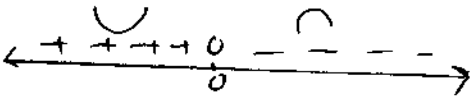
b) Local values:

f(x) has a local min. at $x=-1$ with value $f(-1) = 2 - 3 + 1 = 0$

" " " max. " $x = 1$ " " $f(1) = 2 + 3 - 1 = 4$

c) Concavity / inflection points.

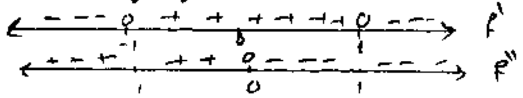
$f''(x) = -6x \overset{set}{=} 0 \Rightarrow x = 0$

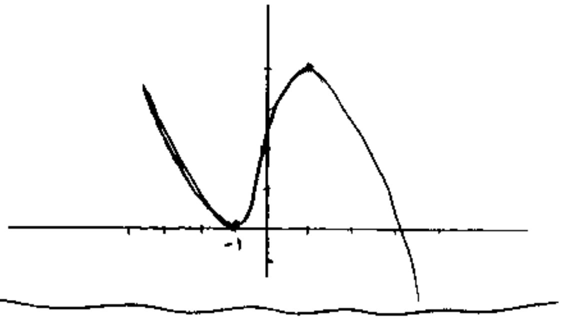


f(x) concave upward on $(-\infty,0)$, concave downward on $(0,\infty)$

$(0, f(0)) = (0,2)$ is the inflection point.

d) The graph of f





Ex.6, 301 $y = x^4 - 4x^3$: Concavity, inflection points, local max. and min. Sketch y.

a) $y' = 4x^3 - 12x^2 \overset{set}{=} 0 \Rightarrow 4x^2(x-3) = 0 \Rightarrow x = 0$, $x = 3$ are critical numbers

$y'' = 12x^2 - 24x \overset{set}{=} 0 \Rightarrow 12x(x-2) = 0 \Rightarrow x = 0, x = 2$

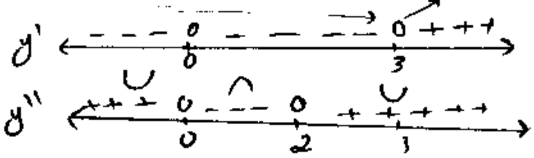
$y''(0) = 0$: The test failed, and we have to use y'.

$y''(3) = 12(3)^2 - 24(3) = 36 > 0 \Rightarrow$ y has a local min. when $x = 3 \Rightarrow (3,-27)$

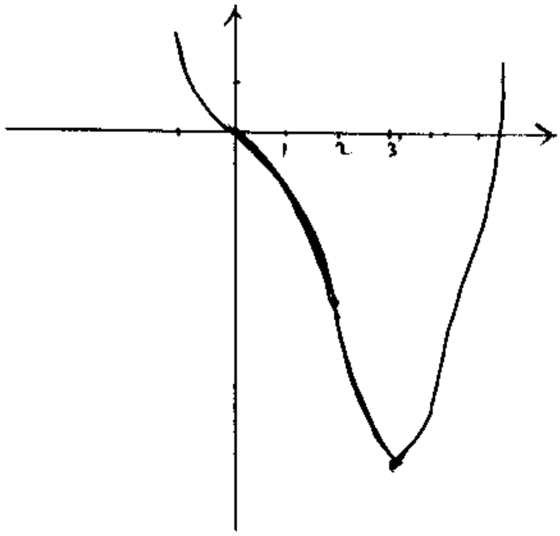
b) y concave upward on $(-\infty,0)$, $(2,\infty)$

" " downward " $(0,2)$

$(0,0)$, $(2,-16)$ are inflection points.

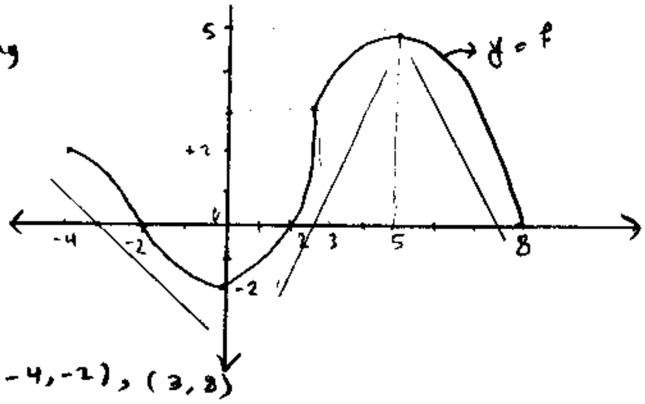


c) The graph of y.



Sec. 4.3

EX, Consider the following graph of f to find the following:



a) The $\overset{\text{open}}{\uparrow}$ intervals on which f is increasing and decreasing.

f is increasing on $(0,5)$
" " decreasing on $(-4,0)$, $(5,8)$

b) The open intervals of concavity

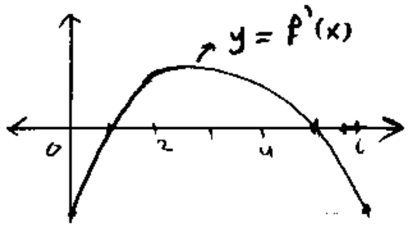
f is concave up on $(-2,3)$ and downward $(-4,-2)$, $(3,8)$

c) Local max. and min. values
$f(0)=-2$ is a local min. and $f(5)=5$ is a local max.

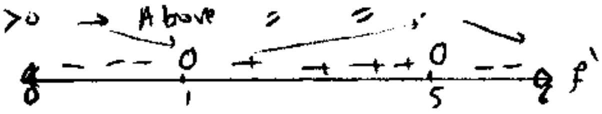
d) Inflection points:
$(-2,0)$ and $(3,3)$ are the inflection points.

Q.5 / **304**, The graph of f' is given
a) on which intervals f is increasing or decreasing?

b) At what values does f has local max. or min.



a
$f'(x)<0 \rightarrow$ below the X-axis
$f'(x)>0 \rightarrow$ Above " " "

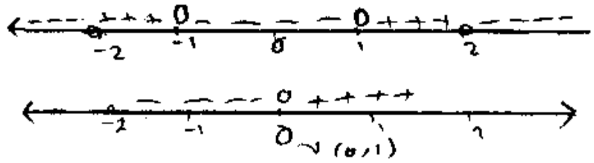


$f(x)$ is increasing on $(1,5)$ and decreasing on $(0,1)$, $(5,6)$

b $f(x)$ has a local min. at $x=1$ and a local max. at $x=5$.

EX, Sketch the graph that satisfies all of the given conditions.

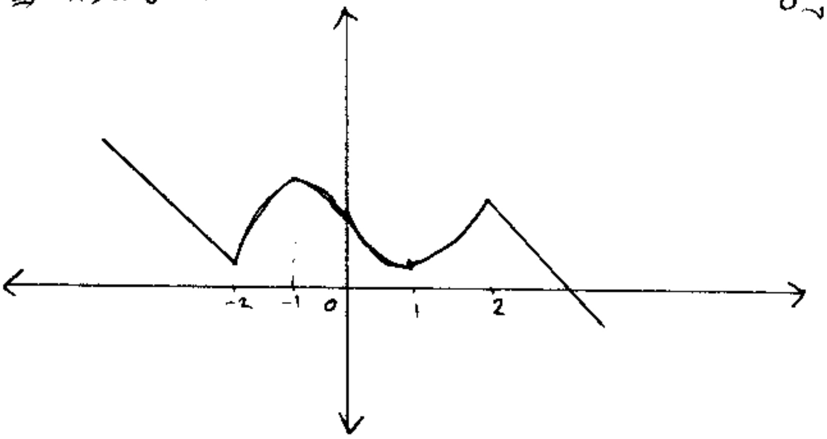
Q.28 / **305**, $f'(1)=f'(-1)=0$, $f'(x)<0$ if $|x|<1$, $f'(x)>0$ if $1<|x|<2$, $f'(x)=-1$ if $|x|>2$
$f''(x)<0$ if $-2<x<0$, inflection point $(0,1)$



$f'(x)<0 \Leftrightarrow -1<x<1$
$f'(x)>0 \rightarrow 1<|x|<2 \Rightarrow -2<x<-1,\ 1<x<2$
$f'(x)=-1$ if $|x|>2 \Rightarrow x>2$ or $x<-2$

one possible graph

is $\Longrightarrow$



Q.45 / 305, $f(x) = \frac{x^2}{x^2-1}$. Dom $f = \mathbb{R} - \{-1,1\} = (-\infty,-1) \cup (-1,1) \cup (1,\infty)$

a) V.A's & H.A's. : V.A's : $x^2-1=0 \Rightarrow x=\pm1$.

$\lim_{x\to-1^-} f(x) = +\infty$, $\lim_{x\to1^+} f(x) = +\infty \Rightarrow x=-1 , x=1$ are V.A's.

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2}{x^2-1} = \lim_{x\to\infty} \frac{1}{1-\frac{1}{x^2}} = 1 \Rightarrow y=1$ is a H.A.

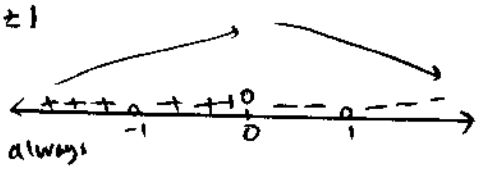
$\lim_{x\to-\infty} f(x) = 1 \Rightarrow y=1$ is a H.A.

---

b) Intervals of increase or decrease

$f'(x) = \frac{(x^2-1)(2x) - x^2(2x)}{(x^2-1)^2} = \frac{-2x}{(x^2-1)^2}$, $x \neq \pm1$

Set $f'(x)=0 \Rightarrow -2x=0 \Rightarrow x=0$, $(x^2-1)^2>0$ always



$f(x)$ is increasing on $(-\infty,-1), (-1,0)$

" " decreasing on $(0,1) , (1,\infty)$

---

c) Local max. and min.

$f(0)=0$ is a local max.

---

d) Concavity and inflection points.

$f''(x) = \frac{(x^2-1)^2(-2) - (-2x)2(x^2-1)(2x)}{(x^2-1)^4} = \frac{2(3x^2+1)}{(x^2-1)^3}$

Set $f''(x)=0 \Rightarrow \underset{>0}{3x^2+1} \neq 0$ No values.

But $f''(x)$ sign depends on denominator

Set $(x^2-1)^3=0 \Rightarrow x^2=1 \Rightarrow x=\pm1$

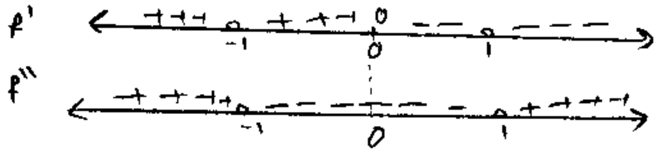
$f(x)$ concave upward on $(-\infty,-1) \cup (1,\infty)$

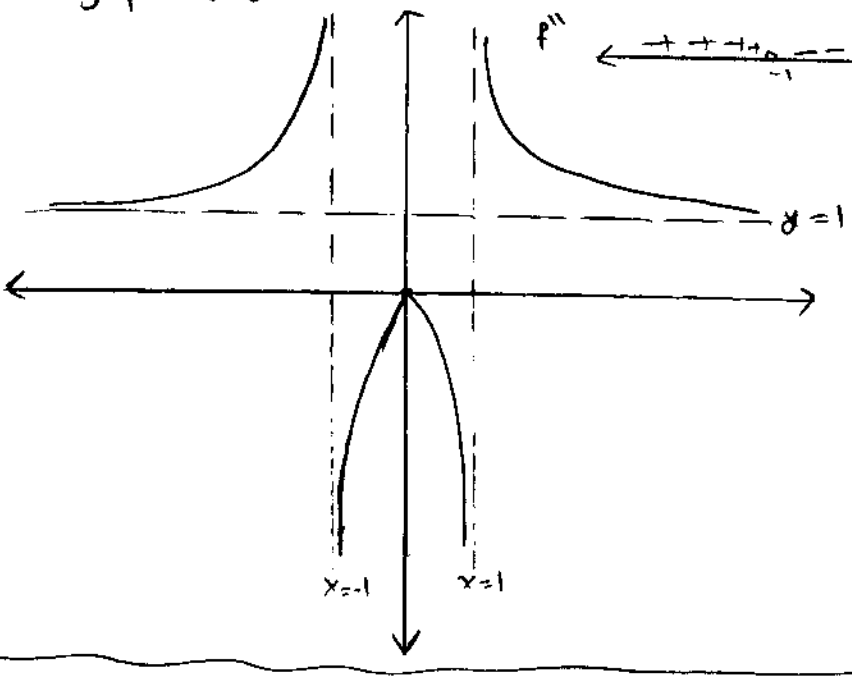
" " downward " $(-1,1)$.

No inflection points.

---

e) Sketch the graph of f





The End.

Sec. 4.4

★Inderminate Forms and L'Hospital's Rule★

objectives: 1. To evaluate indeterminate forms of limits ($\frac{0}{0}$ or $\frac{\infty}{\infty}$) by L'Hopital's Rule.
2. To consider the indeterm : a) Products b) Differences
c) Powers.

L'Hopital's Rule: Suppose f and g are differentiable and $g'(x) \neq 0$ near a (except possibly at a). Suppose that:

$$\lim_{x\to a} f(x) = 0 \quad \text{and} \quad \lim_{x\to a} g(x) = 0 \quad , \text{So} \quad \lim_{x\to a} \frac{f(x)}{g(x} = \frac{0}{0} \text{ type.}$$

or that: $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$ So $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}$ type. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} \quad \text{if exists or } +\infty \text{ or } -\infty.$$

Notes: 1. To use L.R check the limit, and use it only in case of $\frac{0}{0}$ or $\frac{\infty}{\infty}$.
2. L.R. valid as $x \to +$ or $-\infty$ and for one sided limits as $x \to a^-$ or $x \to a^+$.

Ex: Find the limit.

Q-6 / 313: $\lim_{x\to -2} \frac{x+2}{x^2+3x+2} \to \frac{0}{0}$ type

$$= \lim_{x\to -2} \frac{1}{2x+3} = \frac{1}{2(-2)+3} = \frac{1}{-1} = -1$$

H. : Means using L'Hopital's Rule.

Q.10 / 313: $\lim_{x\to 0} \frac{x+\tan x}{\sin x} \to \frac{0}{0}$

$$\overset{H.}{=} \lim_{x\to 0} \frac{1+\sec^2 x}{\cos x} = \frac{1+(1)^2}{1} = \frac{2}{1} = 2.$$

Q-11 / 313: $\lim_{t\to 0} \frac{e^t - 1}{t^3} \to \frac{1-1}{0} = \frac{0}{0}$

$$\lim_{t\to 0} \frac{e^t - 1}{t^3} \overset{H.}{=} \lim_{t\to 0} \frac{e^t}{3t^2} \Rightarrow \lim_{t\to 0^-} \frac{e^t}{3t^2} = \lim_{t\to 0^+} \frac{e^t}{3t^2} = \lim_{t\to 0} \frac{e^t}{3t^2} = \infty$$

Ex.4 / 310: $\lim_{x\to 0} \frac{\tan x - x}{x^3} \to \frac{0}{0}$. Type

$$\overset{H.}{=} \lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2} \to \frac{0}{0} \quad \text{Apply L.R. again.}$$

$$\overset{H.}{=} \lim_{x\to 0} \frac{2\sec x \cdot \sec x \tan x}{3(2x)} = \lim_{x\to 0} \frac{2\sec^2 x \cdot \tan x}{6x}$$

$$= \lim_{x\to 0} \frac{1}{3}\sec^2 x \cdot \lim_{x\to 0} \frac{\tan x}{x} = \frac{1}{3}(1) \cdot \lim_{x\to 0} \frac{\tan x}{x} \to \frac{0}{0} \quad \text{Apply L.R. again}$$

$$\overset{H.}{=} \frac{1}{3} \lim_{x\to 0} \frac{\sec^2 x}{1} = \frac{1}{3}(1) = \frac{1}{3}.$$

Q.23 / 313, $\lim_{x\to\infty} \frac{e^x}{x^3} \to \frac{\infty}{\infty}$ type

$$\overset{H.}{=} \lim_{x\to\infty} \frac{e^x}{3x^2} \to \frac{\infty}{\infty} \quad \text{Apply L.R. again}$$

$$\overset{H.}{=} \lim_{x\to\infty} \frac{e^x}{6x} \to \frac{\infty}{\infty} \quad " \quad " \quad "$$

$$\overset{H.}{=} \lim_{x\to\infty} \frac{e^x}{6} = \infty .$$

Q.31 / 314, $\lim_{x\to\infty} \frac{x}{\ln(1+2e^x)} \to \frac{\infty}{\infty}$ type.

$$\overset{H.}{=} \lim_{x\to\infty} \frac{1}{\frac{2e^x}{1+2e^x}} = \lim_{x\to\infty} \frac{1+2e^x}{2e^x} \to \frac{\infty}{\infty} \quad \text{Apply. L.R. again}$$

$$\overset{H.}{=} \lim_{x\to\infty} \frac{2e^x}{2e^x} = \lim_{x\to\infty} 1 = 1$$

*** Indeterminate Products**

Def, If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$ (or $-\infty$), then $\lim_{x\to a} f(x)g(x)$ is a kind of limits Called indeterminate form of type $0\cdot\infty$.

We can write fg as: $fg = \frac{f}{1/g}$ or $fg = \frac{g}{1/f}$ to convert it to $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and then use L'Hospital's Rule.

Ex.6 / 311, Evaluate $\lim_{x\to 0^+} x \ln x \to (0)(-\infty)$

$$\lim_{x\to 0^+} = \lim_{x\to 0^+} \frac{\ln x}{1/x} \to \frac{\infty}{\infty} \quad \left(\text{or} = \lim_{x\to 0} \frac{x}{\frac{1}{\ln x}} \to \frac{0}{0} \text{ type}\right).$$

$$\overset{H.}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0^+} \frac{1}{x}\cdot\frac{-x^2}{1} = \lim_{x\to 0^+} (-x) = 0 .$$

Q.38 / 314, $\lim_{x\to -\infty} x^2 e^x \to (\infty)(0)$.

$$\lim_{x\to -\infty} x^2 e^x = \lim_{x\to -\infty} \frac{x^2}{e^{-x}} \to \frac{\infty}{\infty}$$

$$\overset{H.}{=} \lim_{x\to -\infty} \frac{2x}{-e^{-x}} \to \frac{\infty}{\infty} \quad \text{Apply L.R. again.}$$

$$\overset{H.}{=} \lim_{x\to -\infty} \frac{2}{e^{-x}} = 0$$

Q.44 / 314, $\lim_{x\to\infty} x \tan(\frac{1}{x}) \to (\infty)(0)$

$$\lim_{x\to\infty} x\tan(\tfrac{1}{x}) = \lim_{x\to\infty} \frac{\tan(\frac{1}{x})}{\frac{1}{x}} \to \frac{0}{0} .$$

$$\overset{H.}{=} \lim_{x\to\infty} \frac{\frac{-1}{x^2}\sec^2(\frac{1}{x})}{\frac{-1}{x^2}} = \lim_{x\to\infty} \sec^2(\tfrac{1}{x})$$

$$= (1)^2 = 1$$

Sec. 4.4

* Indeterminate Differences

Def: If $\lim\limits_{x\to a} f(x) = \infty$ and $\lim\limits_{x\to a} g(x) = \infty$, then

$\lim\limits_{x\to a} [f(x) - g(x)]$ is called an indeterminate form of type $\infty - \infty$

Note, To deal with the form $(\infty - \infty)$, change it to $\frac{\infty}{\infty}$ or $(\frac{0}{0})$ by using a common dinanomator, or ratinalization or factoring out common factor

Ex.7 / 312, Compute $\lim\limits_{x\to \frac{\pi}{2}^-} (\sec x - \tan x) \to \infty - \infty$.

$$\lim_{x\to \frac{\pi}{2}^-} (\sec x - \tan x) = \lim_{x\to \frac{\pi}{2}^-} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right) = \lim_{x\to \frac{\pi}{2}^-} \left(\frac{1-\sin x}{\cos x}\right) \to \frac{0}{0}$$

$$\overset{H.}{=} \lim_{x\to \frac{\pi}{2}^-} \frac{-\cos x}{-\sin x} = \frac{0}{1} = 0 .$$

Q.49 / 314: $\lim\limits_{x\to\infty} (x - \ln x) \to (\infty - \infty)$. As common factor

$$\lim_{x\to\infty} (x - \ln x) = \lim_{x\to\infty} x\left(1 - \frac{\ln x}{x}\right), \text{ but } \lim_{x\to\infty} \frac{\ln x}{x} \overset{H.}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0 .$$

$$= (\infty)(1) = \infty .$$

See Q.50 : Recitation problem.

* Indeterminate Powers

There are several forms of $\lim\limits_{x\to a} [f(x)]^{g(x)}$

1. If $\lim\limits_{x\to a} f(x) = 0$ and $\lim\limits_{x\to a} g(x) = 0$ then it is of type 0^0
2. " $\lim\limits_{x\to a} f(x) = \infty$ " $\lim\limits_{x\to a} g(x) = 0$ " " " " " ∞^0
3. " $\lim\limits_{x\to a} f(x) = 1$ " $\lim\limits_{x\to a} g(x) = \pm\infty$ " " " " " 1^∞

To deal with these limits, there are two suggestions:

a) Taking the natural logarithms $y = (f(x))^{g(x)} \Rightarrow \ln y = g(x) \ln f(x)$

b) Writting the function as an exponential: $[f(x)]^{g(x)} = e^{g(x) \ln f(x)}$

Q.52 / 314, $\lim\limits_{x\to 0^+} (\tan 2x)^x \to 0^0$.

Let $y = (\tan 2x)^x \Rightarrow \ln y = x \ln(\tan 2x)$

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} x \ln(\tan 2x) \overset{(0)(-\infty)}{=} \lim_{x\to 0^+} \frac{\ln(\tan 2x)}{\frac{1}{x}} \overset{H.}{=} \lim_{x\to 0^+} \frac{\frac{2\sec^2(2x)}{\tan 2x}}{\frac{-1}{x^2}}$$

$$= \lim_{x\to 0^+} \frac{\frac{2}{\cos^{\cancel{2}}(2x)} \cdot \frac{\cancel{\cos 2x}}{\sin 2x}}{\frac{-1}{x^2}} = \lim_{x\to 0^+} \frac{2}{\cos(2x)} \cdot \frac{-x^2}{\sin 2x}$$

$$= \lim_{x\to 0^+} \frac{-2x}{\cos 2x} \cdot \lim_{x\to 0^+} \frac{x}{\sin 2x} = (0) \cdot \lim_{x\to 0^+} \frac{1}{2\cos 2x} = (0)(1) = 0$$

$$\Rightarrow \lim_{x\to 0^+} \ln y = 0 \Rightarrow \lim_{x\to 0^+} e^{\ln y} = e^0 \Rightarrow \lim_{x\to 0^+} y = e^0 = 1 .$$

Q.56 / 314: $\lim_{x\to\infty} x^{\frac{\ln 2}{1+\ln x}} \to \infty^0 .$

Let $y = x^{\frac{\ln 2}{(1+\ln x)}} \Rightarrow \ln y = \frac{\ln 2}{1+\ln x} \cdot \ln x$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln 2 \ \ln x}{1+\ln x} \to \frac{\infty}{\infty}$$

$$\overset{H.}{=} \ln 2 . \lim_{x\to\infty} \frac{1 \cancel{\frac{1}{x}}}{\cancel{\frac{1}{x}}} = \ln 2 .$$

$$\therefore \lim_{x\to\infty} y = \lim (x)^{\frac{\ln 2}{1+\ln x}} = e^{\ln 2} = 2.$$

Q.60 / 314: $\lim_{x\to 0} (\cos 3x)^{5/x} \to 1^{\infty}$

Let $y = (\cos 3x)^{\frac{5}{x}} \Rightarrow \ln y = \frac{5}{x} \ln \cos 3x .$

$$\lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{5 \ln \cos 3x}{x} \to \frac{0}{0} .$$

$$\overset{H.}{=} \lim_{x\to 0} \frac{5 \cdot \frac{-3\sin 3x}{\cos 3x}}{1} = -15 \lim_{x\to 0} \frac{\tan 3x}{1} = -15(0) = 0 .$$

$$\Rightarrow \lim_{x\to 0} y = e^{0} = 1 .$$

The End.

Sec. 4.5

* Summary of Curve Sketching *

Objectives, 1. To give guidelines for sketching a curve
2. " Consider the slant asymptotes

*** Guidelines for sketching a curve:**

To sketch the curve $y=f(x)$ by hand, the following guidelines provide all information needed that displays the most important aspects of the function.

1. Find the domain of $f(x)$
2. Find intercepts where For a) The x-intercept: set $y=0$ and solve it for x.
   b) The y-intercept: set $x=0$ and solve it for y.
3. Test the symmetry where:
   (i) If $f(-x)=f(x)$ then f is an even which is symmetric about the y-axis As: $y=x^2$, $y=\cos x$.
   (ii) If $f(-x)=-f(x)$ then $f(x)$ is odd which is symmetric about the origin as $y=x^3$, $y=\sin x$.
   (iii) If $f(x+p)=f(x)$ for all x in Dom. $f(x)$, then f is a periodic function where p is called the period.
   As, $y=\sin x$, $p=2\pi$, $y=\tan x$, $p=\pi$.
4. Find Asymptotes:
   (i) H.A: Find $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$
   (ii) V.A: Find $\lim\limits_{x\to a} f(x)$, when $x=a$ is a V.A.
   (iii) Slant asymptotes (later).
5. Intervals of increasing and decreasing
6. Find local Max. and Min values
7. Find Concavity intervals and inflection point.
8. Use all information to sketch the curve.

Ex. 1 / 318: Sketch $y=f(x)=\dfrac{2x^2}{x^2-1}$

1) Dom $f(x)=\mathbb{R}-\{-1,1\}=(-\infty,-1)\cup(-1,1)\cup(1,\infty)$

2) $f(-x)=\dfrac{2(-x)^2}{(-x)^2-1}=\dfrac{2x^2}{x^2-1}=f(x)$ $\therefore$ even $\Rightarrow f(x)$ is symmetric w.r.t. y-axis.

3) Intercepts: i) x-int.: set $y=0 \Rightarrow 2x^2=0 \Rightarrow x=0 \Rightarrow (0,0)$
2) y-int.: set $x=0 \Rightarrow y=0 \Rightarrow (0,0)$

4) Asymptotes: i) V.A: $\lim_{x\to 1^+} f(x) = \infty$, $\lim_{x\to -1^+} f(x) = -\infty$

$\Rightarrow$ $x=-1$, $x=1$ are V.A's.

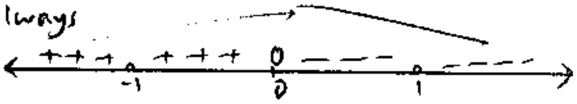
ii) H.A: $\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{2x^2}{x^2-1} = \lim_{x\to\pm\infty} \frac{2}{1-\frac{1}{x^2}} = 2$

$\Rightarrow$ The H.A: $y = 2$.

5) $f'(x) = \frac{(x^2-1)(4x) - 2x^2(2x)}{(x^2-1)^2} = \frac{4x^3-4x-4x^3}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}$.

set $f'(x) = 0 \Rightarrow \frac{-4x}{(x^2-1)^2} = 0 \rightarrow -4x=0 \Rightarrow x=0$.

$(x^2-1)^2 > 0$ always



$f(x)$ is increasing on: $(-\infty, 0)$ and decreasing on $(0,\infty)$

6) $f(x)$ has a local max at $x=0$ with value $f(0) = 0 \Rightarrow (0,0)$.
No local min.

7) $f''(x) = \frac{(x^2-1)^2\cdot(-4) - (-4x)(2(x^2-1))(2x)}{((x^2-1)^2)^2} = \frac{-4(x^2-1)[x^2-1-4x^2]}{(x^2-1)^4}$

$= \frac{-4(-3x^2-1)}{(x^2-1)^3} = \frac{12x^2+4}{(x^2-1)^3}$.

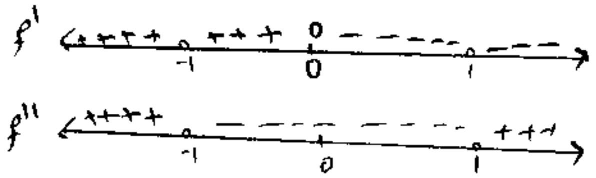
set $f''(x) = 0 \rightarrow \frac{12x^2+4}{(x^2-1)^3} \neq 0$

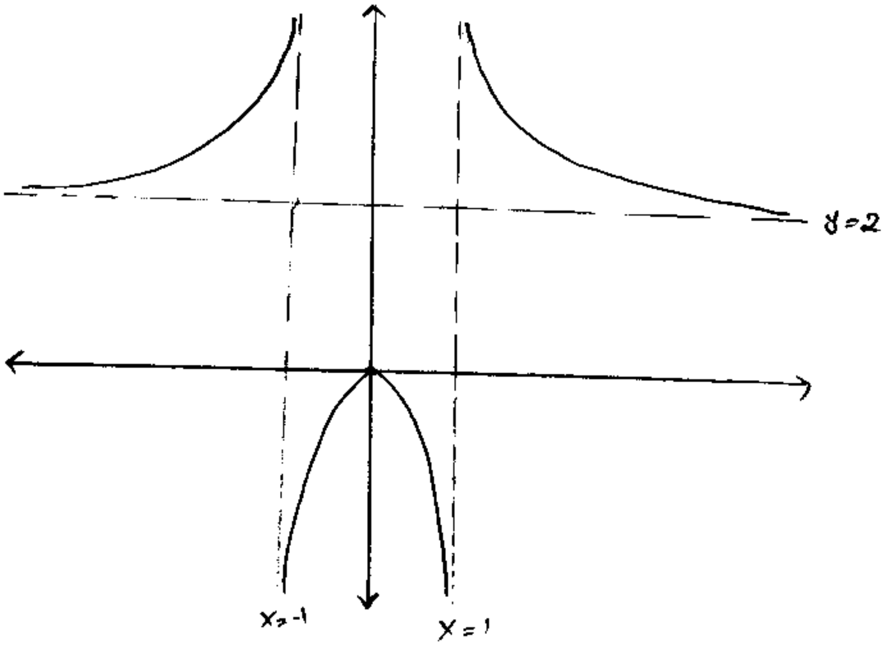
$f''(x)$ depends on $(x^2-1)^3 = 0 \Rightarrow x=\pm 1$

$f(x)$ is concave upward on $(-\infty,-1) \cup (1,\infty)$
" " " downward on $(-1,1)$.
No inflection points.

8).

x	0
y	0





Q.35 / 323 : $y = \frac{1}{2}x - \sin x$, $0 < x < 3\pi$

1) Dom. $= (0, 3\pi)$

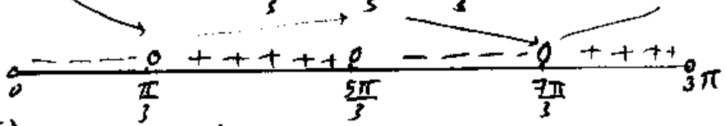
2) i) X-int: $y=0 \Rightarrow \frac{1}{2}x - \sin x = 0 \Rightarrow \frac{1}{2}x = \sin x \Rightarrow x \approx 1.9 \rightarrow (1.9, 0)$ using Newton's Method.
   ii) Y-int: $x = 0 \Rightarrow y = 0 - 0 = 0 \Rightarrow (0,0)$

3) $f(-x) = -\frac{1}{2}x + \sin x = -(\frac{1}{2}x - \sin x) = -f(x)$ symm. about origin.

4) No asymptote.

5) $f'(x) = \frac{1}{2} - \cos x \Rightarrow f'(x) = 0 \Rightarrow \cos x = \frac{1}{2} \Rightarrow x = \frac{\pi}{3}, 2\pi - \frac{\pi}{3}, 2\pi + \frac{\pi}{3}$

$\Rightarrow x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$.



f(x) is increasing on $(\frac{\pi}{3}, \frac{5\pi}{3}), (\frac{7\pi}{3}, 3\pi)$
" " decreasing on $(0, \frac{\pi}{3}), (\frac{5\pi}{3}, \frac{7\pi}{3})$

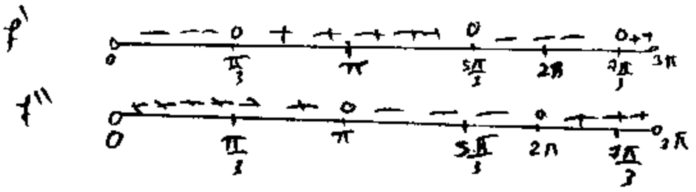
6) f has a local min. at $x = \frac{\pi}{3} \rightarrow f(\frac{\pi}{3}) = \frac{\pi}{6} - \frac{\sqrt{3}}{2} \approx -.34 \rightarrow (1.05, -.34)$
" " " " at $x = \frac{7\pi}{3} \rightarrow f(\frac{7\pi}{3}) = \frac{7\pi}{6} - \frac{\sqrt{3}}{2} \approx 2.8 \rightarrow (7.33, 2.8)$
" " " max. at $x = \frac{5\pi}{3} \rightarrow f(\frac{5\pi}{3}) = \frac{5\pi}{6} + \frac{\sqrt{3}}{2} \approx 3.48 \rightarrow$

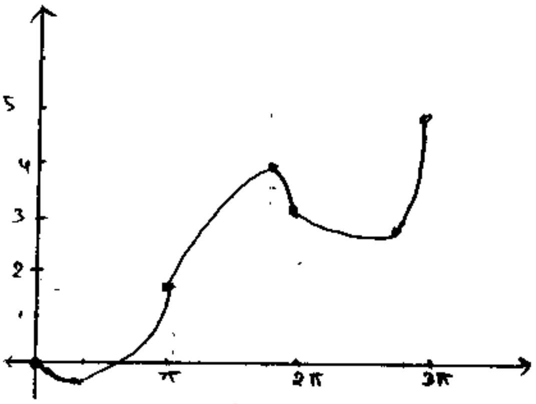
7) $f''(x) = \sin x$
set $f''(x) = 0 \Rightarrow \sin x = 0 \Rightarrow x = \pi, 2\pi$

f is concave up on $(0, \pi), (2\pi, 3\pi)$
" " " down " $(\pi, 2\pi)$

$(\pi, f(\pi)) = (\underset{3.14}{\pi}, \underset{1.57}{\frac{\pi}{2}})$, $(2\pi, f(2\pi)) = (\underset{6.28}{2\pi}, \underset{3.14}{\pi})$

X	0	π/3	π	5π/3	2π	7π/3	3π
y	0	-.34	$\frac{\pi}{2}$ ≈ 1.57	3.48	π 3.14	2.8	$\frac{3\pi}{2}$ ≈ 4.7





✱ Slant Asymptote ; It is neither horizontal nor vertical (oblique).

Def: If $\lim_{x \to \pm\infty} [f(x) - (mx+b)] = 0$ then $y = mx+b$ is a slant asymptote.

Q.55 / 324 : Find the equ. of the slant asymptote $y = \frac{x^2+1}{x+1}$

$$\begin{array}{r} x-1 \\ x+1 \overline{) x^2 + 1} \\ \mp x^2 \mp x \\ \hline -x + 1 \\ \pm x \pm 1 \\ \hline 2 \end{array}$$

$y = x - 1 + \frac{2}{x+1} \Rightarrow y = x-1$ is a S.A.

note that $f(x) - (x-1) = x - 1 + \frac{2}{x-1} - (x-1) = \frac{2}{x-1}$

$\lim_{x \to \pm\infty} f(x) - y = \lim_{x \to \infty} \frac{2}{x-1} = 0$

Q.10 / 324, Sketch $y = f(x) = \frac{x^2+12}{x-2}$.

① DOM. $= \mathbb{R} - \{2\} = (-\infty, 2) \cup (2, \infty)$

② i) X-Int: Set $y=0 \Rightarrow \frac{x^2+12}{x-2} \neq 0$ No X-int.

ii) Y-int: Set $x=0 \Rightarrow y = \frac{12}{-2} = -6 \Rightarrow (0,-6)$

③ $f(-x) = \frac{x^2+12}{-x-2} \neq f(x)$ or $-f(x)$: Not symm.

④ i) V.A: $x=2$ is a V.A because $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{x^2+12}{x-2} = +\infty$.

ii) H.A: $\lim_{x\to\infty} f(x) \overset{H.}{=} \lim_{x\to\infty} \frac{2x}{1} = +\infty$

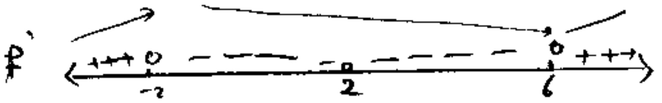
$\lim_{x\to-\infty} f(x) \overset{H.}{=} \lim_{x\to-\infty} \frac{2x}{1} = -\infty \Longrightarrow$ NO H.A.

$$\begin{array}{r|l} & x+2 \\ \hline x-2 & \cancel{x^2}+12 \\ & \mp x^2 \pm 2x \\ \hline & \cancel{2x}+12 \\ & \mp 2x \pm 4 \\ \hline & 16 \end{array}$$

iii) S.A: $y = x+2+\frac{16}{x-2} \Rightarrow y = x+2$ is a S.A.

Because $\lim_{x\to\pm\infty} [f(x)-(x+2)] = \lim_{x\to\pm\infty} \frac{16}{x-2} = 0$.

⑤ $f'(x) = \frac{(x-6)(x+2)}{(x-2)^2} \overset{\text{Set}}{=} 0 \Rightarrow x=6, x=-2$ and $(x-2)^2 > 0$ always.



f is increasing on : $(-\infty,-2)$, $(6,\infty)$ and decreasing on $(-2,2)$, $(2,6)$

⑥ f has a local max. at $-2 \to (-2, f(-2)) = (-2,-4)$
" " " Min. " $6 \to (6, f(6)) = (6,12)$.

⑦ $f''(x) = \frac{32}{(x-2)^3} \overset{\text{Set}}{=} 0$ No Solutn. But $(x-2)^3 = 0 \Rightarrow x=2$

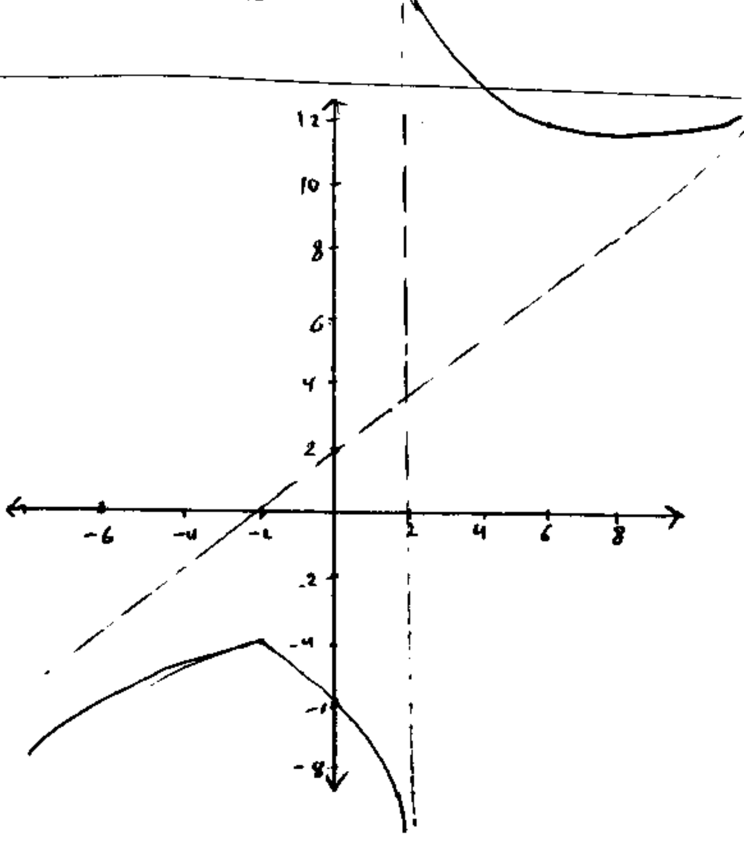
Concave upward on $(2,\infty)$
" downward on $(-\infty,2)$
No inflection points.

⑧

x	-2	0	2	6
y	-4	-6	V.A	12

$y = x+2$

x	-2	0
y	0	2



The End.

Sec. 4.7

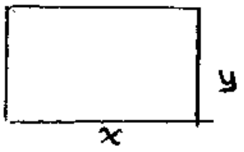
Optimization Problems

Objectives : To solve practical applications in optimization

Steps in Solving optimization problems : See text page 331-332.

Q-5, **336** Let the dimensions be x, y.

" " area be A



$A = xy$ but perimeter $= 100$

$2x + 2y = 100 \Rightarrow x + y = 50 \Rightarrow y = 50 - x$, $0 < x < 50$

$A = x(50 - x)$ $\qquad A(0) = A(50) = 0$

$A = 50x - x^2$

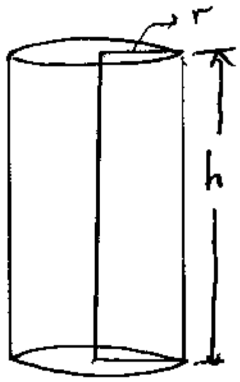
$A' = 50 - 2x \rightarrow$ set $A' = 0 \Rightarrow 50 = 2x \Rightarrow x = 25\,m$

$A'' = -2 \Rightarrow A''\Big|_{x=25} = -2 < 0 \Rightarrow$ Max. when $x = 25$.

$\Rightarrow$ The dimensions are : $x = 25\,m$, $y = 50 - 25 = 25\,m$.

Ex. 2, **333** 1 L of oil. Minimize the Cost. $1L = 1000\ cm^3$.

Let the radius be r, h the height, C : cost.



$C =$ Cost of surface area $+$ Cost of bases.

$= 2\pi r^2 + 2\pi r h$ But $\pi r^2 h = 1000$

$\Rightarrow h = \frac{1000}{\pi r^2}$.

$C = 2\pi r^2 + 2\cancel{\pi} r \cdot \frac{1000}{\cancel{\pi r^2}}$

$= 2\pi r^2 + \frac{2000}{r}$. $\quad , r > 0$

$C' = 4\pi r - \frac{2000}{r^2} \overset{\text{set}}{=} 0 \rightarrow \frac{4\pi r}{1} = \frac{2000}{r^2}$

$4\pi r^3 = 2000 \Rightarrow r^3 = \frac{2000}{4\pi} = \frac{500}{\pi}$

$\Rightarrow r = \sqrt[3]{\frac{500}{\pi}}$

$C'' = 4\pi + \frac{2000(2r)}{r^4} \Rightarrow C''\Big|_{r=\sqrt[3]{\frac{500}{\pi}}} > 0$

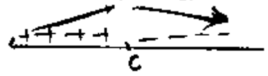
$\Rightarrow$ Min. value when $r = \sqrt[3]{\frac{500}{\pi}}$

$h = \frac{1000}{\pi\left(\left(\frac{500}{\pi}\right)^{\frac{1}{3}}\right)^2} = \frac{1000}{\pi\left(\frac{500}{\pi}\right)^{2/3}} = 2\sqrt[3]{\frac{500}{\pi}} = 2r$

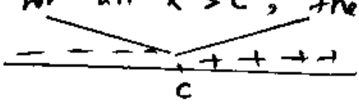
First Derivative Test for Absolute Extreme Values ;

Suppose c is a critical of a conts. function f on an interval

a. If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then $f(c)$ is the absolute max. of f



b. If $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, then $f(c)$ is the absolute min. of f



Q.16 / 337: Find the point on $6x+y=9$ that is closest to $(-3,1)$
$y=9-6x$.

Let (x,y) be the point on the line.
D " " distance between (x,y) and $(-3,1)$

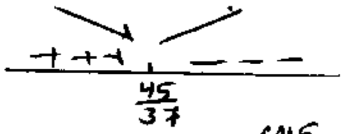
$$D=\sqrt{(x+3)^2+(y-1)^2} \quad \text{But} \quad y=9-6x.$$

$$D=\sqrt{(x+3)^2+(9-6x-1)^2}=\sqrt{(x+3)^2+(8-6x)^2}$$

$$D'=\frac{2(x+3)+2(8-6x)(-6)}{2\sqrt{(x+3)^2+(8-6x)^2}} \overset{\text{set}}{=} 0 \Rightarrow 2(x+3)+2(8-6x)(-6)=0$$

$$\Rightarrow x+3+(8-6x)(-6)=0 \Rightarrow x+3+36x-48=0$$

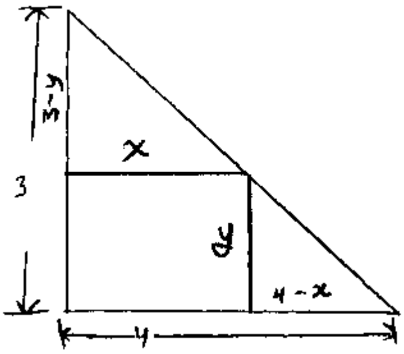
$$37x+45=0 \Rightarrow x=\frac{45}{37}.$$



∴ Min. when $x=\frac{45}{37}$

$$y=9-6x=9-6\left(\frac{45}{37}\right)=\frac{63}{37} \Rightarrow \text{The point } \left(\frac{45}{37},\frac{63}{37}\right)$$

Q.24 / 337: Let the dimensions be x and y.
Let A be the area:



$A=xy$. But by simillar triangles

$$\frac{3-y}{x}=\frac{y}{4-x}$$

$$xy=12-3x-4y+xy$$

$$4y=12-3x \Rightarrow y=3-\frac{3}{4}x$$

OR : $\frac{3-y}{x}=\frac{3}{4}$

$12-4y=3x$

$\Rightarrow y=3-\frac{3}{4}x$

$$A=x\left(3-\frac{3}{4}x\right)=3x-\frac{3x^2}{4}, \quad 0\le x\le 4. \quad A(0)=A(4)=0.$$

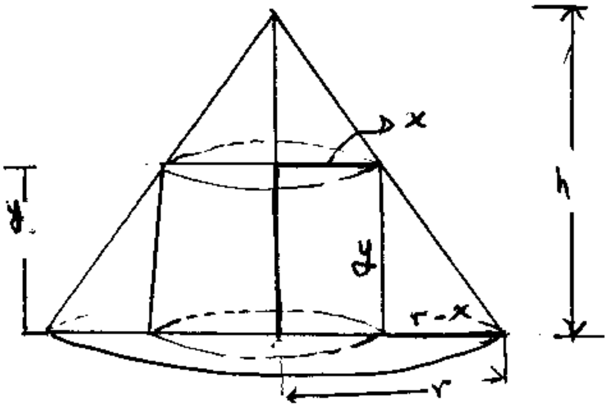
$$A'=3-\frac{3x}{2} \overset{\text{set}}{=} 0 \Rightarrow 3=\frac{3x}{2} \Rightarrow x=2.$$

$$A''=\frac{-3}{2} \Rightarrow A''\big|_{x=2}=-\frac{3}{2}<0 \quad \Rightarrow \quad \text{Max. when } x=2 \text{ cm}$$

The dimensions are : $x=2$, $y=3-\frac{3}{4}(2)=3-\frac{3}{2}=\frac{3}{2}$ cm

Max. area $=(2)\left(\frac{3}{2}\right)=3$ cm².

Q.26 / 337: Let x be the radius of the Cylinder
y " " hight " " "
V " " volume " " "



$V=\pi x^2 y$ By simillar triangles

$$\frac{y}{r-x}=\frac{h}{r}$$

$$\Rightarrow y=\frac{hr-xh}{r}=h-\frac{h}{r}x$$

$$=h\left(1-\frac{x}{r}\right)$$

$$V=\pi x^2\left(h\left(1-\frac{x}{r}\right)\right)=h\pi\left(x^2-\frac{1}{r}x^3\right)$$

$$V'=h\pi\left(2x-\frac{3}{r}x^2\right) \overset{\text{set}}{=} 0 \Rightarrow 2x-\frac{3}{r}x^2=0 \Rightarrow x\left(2-\frac{3x}{r}\right)=0 \Rightarrow x=0$$

or $2=\frac{3x}{r} \Rightarrow x=\frac{2}{3}r$.

$$V''=h\pi\left(2-\frac{6}{r}x\right) \Rightarrow V''\big|_{x=\frac{2}{3}r}<0 \Rightarrow \text{Max. value when } x=\frac{2}{3}r$$

∴ The largest possible voluem $=V\left(\frac{2}{3}r\right)=h\pi\left(\frac{4}{9}r^2-\frac{1}{r}\cdot\frac{8}{27}r^3\right)=\frac{4}{27}\pi r^2 h.$

The End

Sec. 4.9

Newton's Method

objective. To find a certain approximation of a root for an equation using Newton's method

---

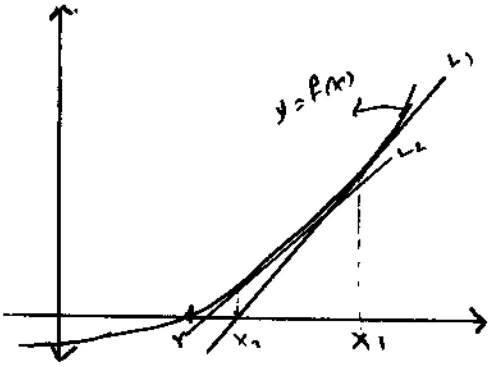
Consider $y = f(x)$.

The root of $y = f(x)$ is r.

There are many methods to find r.

We will use a method called Newton's Method

or Newton-Raphson method.



N-R. method.

1. Start with initial approximation say x_1.
2. Find the tangent at x_1: $y - f(x_1) = f'(x_1)(x - x_1)$
3. Find = intersection with the x-axis (x-intercept) of the tangent

Set $y = 0 \Rightarrow -f(x_1) = f'(x_1)(x - x_1)$, the x-int. is x_2.

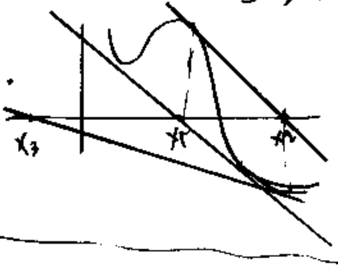
$$\Rightarrow -f(x_1) = f'(x_1)(x_2 - x_1)$$

$$\Rightarrow x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

4. Repeat the process to get $x_1, x_2, x_3, \dots$ and in general $\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$

Note 1. If x_n become closer and closer to r as n becomes large, then $\lim_{n \to \infty} x_n = r$.

2. The N-M fails if the intial guess is bad.
(Very sensitive to the intial guess).



Q.6 / 351. Use N-M to find x_3.

$x^3 - x^2 - 1 = 0$, $x_1 = 1$.

Let $f(x) = x^3 - x^2 - 1$, $x_1 = 1$, $f'(x) = 3x^2 - 2x$.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \Rightarrow x_2 = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{-1}{1} = 1+1 = 2$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \Rightarrow x_3 = 2 - \frac{3}{8} = \frac{13}{8} = 1.625.$$

Generaly: $x_{n+1} = x_n - \dfrac{x_n^3 - x_n^2 - 1}{3x_n^2 - 2x_n}$.

Q.7 / 351. $x^4 - 20 = 0$, $x_1 = 2$

Let $f(x) = x^4 - 20 \Rightarrow f'(x) = 4x^3$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^4 - 20}{4x_n^3}.$$

$$x_2 = 2 - \frac{2^4 - 20}{4(2)^3} = 2 - \frac{(-4)}{32} = 2 + \frac{1}{8} = 2.125.$$

$$x_3 = 2.125 - \frac{(2.125)^4 - 20}{4(2.125)^3} = 2.1148.$$

Sec. 4.9

$\frac{\text{Ex. 2}}{350}$: Use N-M to find $\sqrt[6]{2}$ correct to eight decimal places: let $x = \sqrt[6]{2}$

$x^6 = 2$

$x^6 - 2 = 0$

Let $f(x) = x^6 - 2$, $f'(x) = 6x^5$.

$$X_{n+1} = X_n - \frac{f(X_n)}{f'(X_n)} \Rightarrow X_{n+1} = X_n - \frac{X_n^6 - 2}{6X_n^5}.$$

Let $X_1 = 1 \Rightarrow X_2 = 1 - \frac{1-2}{6} = 1 - \frac{(-1)}{6} = 1 + \frac{1}{6} = 1.1666667$

$X_3 \approx 1.12644368$, $X_5 \approx 1.12246205$

$X_4 \approx 1.12249707$, $X_6 \approx 1.12246205$. $\Big\} \rightarrow x_5 - x_6 = 0$

$\therefore \sqrt[6]{2} \approx 1.12246205$.

$\frac{\text{Q.15}}{351}$: Use N-M to approximate the indicated root to six decimal places

$\sin X = x^2$. $\Rightarrow \sin x - x^2 = 0$

Let $f(x) = \sin x - x^2 \Rightarrow f'(x) = \cos x - 2x$.

Let $\boxed{X_1 = 1 \text{ (in rad)}}$ $\quad X_{n+1} = X_n - \frac{\sin(X_n) - X_n^2}{\cos(X_n) - 2X_n}$.

$$X_2 \approx 1 - \frac{\sin(1) - (1)^2}{\cos(1) - 2} = 1 - 0.108604 = 0.891396$$

$$X_3 \approx 0.891396 - \frac{\sin(.891396) - (.891396)^2}{\cos(.891396) - 2(.891396)}$$

$X_4 \approx 0.876985$

$X_5 \approx 0.876726$

$X_6 \approx 0.876726$

We see that there is no difference between X_5 & X_6 to six decimal places.

$\therefore$ Root ≈ 0.876726

The End

Sec. 4.10

* Antiderivatives *

objective, 1. To define the antiderivative
2. " " differential equation and solve physical applications

Def, A function F is called an antiderivative of f on an interval I if
$F'(x) = f(x)$ for all x in I.

Ex, Let $f(x) = x^2$ then:
$F(x) = \frac{1}{3}x^3$ is an antiderivative becau $F'(x) = x^2$
$G(x) = \frac{1}{3}x^3 + 100$, " " " $G'(x) = x^2$. $(G(x) - F(x) = c)$
⋮
In general $\frac{1}{3}x^3 + c$ is " " "

Def, If F(x) is an antiderivative of f on an interval I, then the most general antiderivative of f on I is: $F(x) + c$, c is constant.

Ex, Find F(x) if $f(x) = x^n$, $n \neq -1$

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n \Rightarrow F(x) = \frac{x^{n+1}}{n+1} + c$$ is an antiderivative.

Table of Antidifferentiation Formulas (see page 354)

Func.	Anti der	Fun.	Anti.
$c\,f(x)$	$c\,F(x)$	$\sin x$	$-\cos x$
$f(x) + g(x)$	$F(x) + G(x)$	$\sec^2 x$	$\tan x$
$x^n \, (n \neq -1)$	$\frac{x^{n+1}}{n+1}$	$\sec x \tan x$	$\sec x$
$\frac{1}{x}$	$\ln \lvert x \rvert$	$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$
e^x	e^x	$\frac{1}{1+x^2}$	$\tan^{-1} x$.
$\cos x$	$\sin x$		

Ex, Find the antiderivative

$\frac{Q.2}{358}$, $f(x) = 4 + x^2 - 5x^3 \Rightarrow F(x) = 4x + \frac{1}{3}x^3 - \frac{5}{4}x^4 + C$.

$\frac{Q.10}{358}$, $g(x) = \frac{5 - 4x^3 + 2x^6}{x^6} = \frac{5}{x^6} - \frac{4}{x^3} + 2 = 5x^{-6} - 4x^{-3} + 2$, $x \neq 0$
$G(x) = \frac{5x^{-5}}{-5} - \frac{4x^{-2}}{-2} + 2x + c = -x^{-5} + 2x^{-2} + 2x + C = \frac{-1}{x^5} + \frac{2}{x^2} + C$ for $x \neq 0$

$\frac{Q.12}{358}$, $f(x) = 3e^x + 7\sec^2\theta \Rightarrow F(x) = 3e^x + 7\tan\theta + C$.

$\frac{Ex.2}{355}$, Find g such that $g'(x) = 4\sin x + \frac{2x^5 - \sqrt{x}}{x} \rightarrow g'(x) = 4\sin x + 2x^4 - x^{-\frac{1}{2}}$
$G(x) = -4\cos x + \frac{2}{5}x^5 - \frac{x^{1/2}}{1/2} + c$

$g(x) = -4\cos x + \frac{2}{5}x^5 - 2\sqrt{x} + C$

Def, Differential equation : It is an equation involves the derivatives of a function.

EX, Find f

Q.22 / **358**, $f''(x) = \cos x$

$f'(x) = \sin x + c$

$f(x) = -\cos x + cx + D$.

Q.26 / **358**, $f'(x) = 8x^3 + 12x + 3$, $f(1) = 6$ (It is called initial value problem)

$f(x) = \frac{8x^4}{4} + \frac{12x^2}{2} + 3x + C = 2x^4 + 6x^2 + 3x + C$

$f(1) = 2 + 6 + 3 + C = 6 \Rightarrow C = -5$, So $f(x) = 2x^4 + 6x^2 + 3x - 5$

Q.35 / **358**, $f''(\theta) = \sin\theta + \cos\theta$. $f(0) = 3$, $f'(0) = 4$.

$f'(\theta) = -\cos\theta + \sin\theta + C \Rightarrow f'(0) = -1 + 0 + C = 4 \Rightarrow C = 5$.

So $f'(\theta) = -\cos\theta + \sin\theta + 5$

$f(\theta) = -\sin\theta - \cos\theta + 5\theta + D$ but $f(0) = 3$

$\Rightarrow f(0) = 0 - 1 + 0 + D = 3 \Rightarrow D = 4$ So : $f(\theta) = -\sin\theta - \cos\theta + 5\theta + 4$

Q.40 / **359**, $f''(t) = 2e^t + 3\sin t$, $f(0) = 0$, $f(\pi) = 0$

$f'(t) = 2e^t + 3(-\cos t) + C = 2e^t - 3\cos t + C$

$f(t) = 2e^t - 3\sin t + ct + D$

But $f(0) = 0 \Rightarrow 2 - 0 + 0 + D = 0 \Rightarrow D = -2$

$f(\pi) = 0 \Rightarrow 2e^{\pi} - 0 + c\pi - 2 = 0 \Rightarrow c\pi = 2 - 2e^{\pi} \Rightarrow c = \frac{2 - 2e^{\pi}}{\pi}$

$\therefore f(t) = 2e^t - 3\sin t + \frac{2 - 2e^{\pi}}{\pi} t - 2$.

*** Rectlinear Motion**, Recall that if $S = f(t)$ the position function, then the Velocity is $v(t) = S' = f'(t)$ and the acceleration $a(t) = v'(t) = f''(t)$.

EX, A particle is moving with the given data. Find the position of the particle.

Q.59 / **359**, $v(t) = \sin t - \cos t$, $S(0) = 0$

$S(t) = -\cos t - \sin t + C$, $S(0) = -1 - 0 + C = 0 \Rightarrow C = 1$

$\Rightarrow$ The position func. is $S(t) = -\cos t - \sin t + 1$.

Q.64 / **360**, $a(t) = 10 + 3t - 3t^2$, $S(0) = 0$, $S(2) = 10$

$v(t) = 10t + \frac{3}{2}t^2 - t^3 + C$

$S(t) = \frac{10t^2}{2} + \frac{3}{2} \cdot \frac{t^3}{3} - \frac{t^4}{4} + ct + D = 5t^2 + \frac{1}{2}t^3 - \frac{1}{4}t^4 + Ct + D$

$S(0) = 0 \Rightarrow D = 0$

$S(2) = 10 \Rightarrow 20 + 4 - 4 + 2C = 10 \Rightarrow 2C = -10 \Rightarrow C = -5$

So : $S(t) = 5t^2 + \frac{1}{2}t^3 - \frac{1}{4}t^4 - 5t$

---

* The End of Math 101 NOTES *