TOEPLITZ MATRIX APPROXIMATION

by

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1 Matrix nearness problems

Given a matrix $F \in \mathbb{R}^{n\times n}$ **then consider the problem**

$$\underset{\text{D}}{\text{minimize}} \ \|F - D\| \qquad (\|F - D\| \leq \epsilon|F|)$$

Such that D **has property P**

P can be any one or mixture of:

* **Symmetry**

* **Skew-Symmetry**

* **Poitive semi-definiteness**

* **Orthogonality, Unitary**

* **Normality**

* **Rank-deficiency, Singularity**

* D **in a linesr space**

* D **with some fixed columns, rows, submatrix**

* **Instability**

* D **with a given** λ**, repeated** λ

* D **is Euclidean Distance Matrix**

* D **is Toeplitz or Hankel**

2 Hybrid Methods for Finding the Nearest Euclidean Distance Matrix

Definition

A matrix $D \in \mathbb{R}^{n\times n}$ is called a Euclidean distance matrix iff there exist n points $\mathbf{x}_1, \ldots, \mathbf{x}_n$ in an affine subspace of dimension $\mathbb{R}^m$ $(m \leq n-1)$ such that

$$d_{ij} = \|\mathbf{x}_i - \mathbf{x}_j\|_2^2 \qquad \forall i, j. \tag{1}$$

The Euclidean distance problem can now be stated as follows. Given a matrix $F \in \mathbb{R}^{n\times n}$, find the Euclidean distance matrix $D \in \mathbb{R}^{n\times n}$ that minimizes

$$\|F - D\|_F \tag{2}$$

where $\|.\|_F$ denotes the Frobenius norm. see Al-Homidan and Fletcher [1]

3 Educational Testing Problem

The educational testing problem. can be expressed as

$$\begin{array}{lllll} \text{maximize} & \mathbf{e}^T\boldsymbol{\theta} & & \boldsymbol{\theta} \in \mathbb{R}^n & \\ \text{subject to} & F - \operatorname{diag} \boldsymbol{\theta} \geq 0 & & & \\ & \theta_i \geq 0 & & i = 1, ..., n & (3) \end{array}$$

where $\mathbf{e} = (1, 1, ..., 1)^T$**. An equivalent form of (3) is**

$$\begin{array}{lllll} \text{minimize} & \mathbf{e}^T\mathbf{x} & & \mathbf{x} \in \mathbb{R}^n & \\ \text{subject to} & \bar{F} + \operatorname{diag} \mathbf{x} \geq 0 & & & \\ & x_i \leq v_i & & i = 1, ..., n & (4) \end{array}$$

where $\bar{F} = F - \operatorname{Diag} F$ **and** $\operatorname{diag} \mathbf{v} = \operatorname{Diag} F$.
Where this problem is expressed later as matrix nearness approximated to a matrix satisfy certain conditions.
see Al-Homidan [2]

4 Hybrid Methods for Minimizing Least Distance Functions with Semi-Definite Matrix Constraints

We are interested here in problems in which only the diagonal of the matrix is allowed to change, in the following way. Given a symmetric positive definite matrix $F \in \mathbb{R}^{n\times n}$ then we consider the problem

$$\begin{array}{llll} \text{minimize} & \|\mathbf{a} - \mathbf{x}\|_2^2 & \mathbf{x} \in \mathbb{R}^n & \\ \text{subject to} & \bar{F} + \operatorname{diag}\mathbf{x} \geq 0, & \mathbf{x} \leq \mathbf{v} & (5) \end{array}$$

where a is an initial point and then we have a different problem with every different a.

Also this problem is expressed later as matrix nearness approximated to a matrix satisfy certain conditions.

see Al-Homidan [3]

5 The Problem

The problem we are interested in is the best approximation of a given matrix D by a positive semidefinite symmetric Toeplitz matrix. Related problems occur in many engineering and staistical applications [4], especially in the area of signal processing. Because of rounding errors or truncation errors incurred when evaluating F, F does not satisfy one or all conditions. Toeplitz matrix approximation are discussed in [6],[9] and [5]

We consider the following problem: Given a data matrix $F \in \mathbb{R}^{n\times n}$ find the nearest symmetric positive semi-definite toeplitz matrix D to F. Use of the Frobenius norm as a measure

gives rise to the problem

$$\begin{aligned} &\text{minimize} \quad \Phi = \|F \;-\; D\| \\ &\text{subject to} \quad D \;\in\; K. \end{aligned} \tag{6}$$

where K is the set of all $n \times n$ symmetric positive semi-definite toeplitz matrices

$$K = \{A : A \in \mathbb{R}^{n\times n}, A^T = A, \;\; A \geq 0 \;\; \text{and} \;\; A \;\in\; T\} \tag{7}$$

where T the set of all toeplitz matrices.

The problem is formulated as a nonlinear minimization problem, with positive semi-definite toeplitz matrix as constraints. Then a computational framework is given. An algorithm with rapid convergence is obtained by l_1Sequential Quadratic Programming method.

Theorem

Problem(6) has a unique solution for rank F $m = n$ or $m = n - 1$ if the data matrix in not positive semi-definite. In all other cases there exists a solution which may not be unique.

6 l_1SQP Method

This section contains a brief description of the l_1SQP method for solving (6).

It is difficult to deal with the matrix cone constraints in (7) since it is not easy to specify if the elements are feasible or not. Using partial LDL^T factorization of A, this difficulty is rectified. Since m, the rank of A^*, is known, then for A sufficiently close to A^*, the partial

factors $A = LDL^T$ **can be calculated where** $L = \begin{bmatrix} L_{11} & \\ L_{21} & I \end{bmatrix}, D = \begin{bmatrix} D_1 & \\ & D_2 \end{bmatrix}, A = \begin{bmatrix} A_{11} & A_{21}^T \\ A_{21} & A_{22} \end{bmatrix}$. **where** $L_{11},\ D_1$ **and** A_{11}**are** $m \times m$ **matrices,** $I,\ D_2$ **and** A_{22}**are** $n-m \times n-m$ **matrices,** L_{21} **and** A_{21} **are** $n-m \times m$ **matrices, and** D_2 **has no particular structure other than symmetry. At the solution** $D_2 = 0$**. In general**

$$D_2(A) \ = \ A_{22} \ - \ A_{21}A_{11}^{-1}A_{21}^T, \tag{8}$$

this expression enables the constraint $D \in k$ **to be written in the form**

$$D_2(D) = 0 \tag{9}$$

Then problem (6) can be expressed as

$$\begin{aligned} &\text{minimize} \quad \Phi \\ &\text{subject to} \quad D_2(D) \ = \ 0 = Z^T D Z, \end{aligned} \tag{10}$$

where

$$Z = \begin{bmatrix} -A_{11}^{-1}A_{21}^{T} \\ I \end{bmatrix}$$

the basis matrix for the null space of D **when** $D_2 = 0$**. The Lagrange multipliers for the constraint (9) is** Λ **relative to the basis** Z **and the Lagrangian for porblem (10) is**

$$\mathcal{L}(\mathbf{x}^{(k)}, \Lambda^{(k)}, \boldsymbol{\pi}^{(k)}) = \Phi - \Lambda : Z^T D Z \qquad (11)$$

Since D **is toeplitz matrix the** D **have the following structure**

$$D = \begin{bmatrix} x_1 & \cdots & x_n \\ \vdots & \ddots & \vdots \\ x_n & \cdots & x_1 \end{bmatrix} \qquad (12)$$

then

$$\begin{aligned} \Phi &= \sum_{i,j=1}^{n} (f_{ij} - d_{ij})^2 \\ &= \sum_{i,j=1}^{n} (f_{ij} - x_{|i-j+1|})^2. \end{aligned} \qquad (13)$$

and

$$\bigtriangledown\Phi \;=\; [\frac{\partial\Phi}{\partial x_1}\cdots\frac{\partial\Phi}{\partial x_n}]^2$$

where ∇ **denotes the gradient operator** $(\partial/\partial x_1, \ \ldots, \ \partial/\partial x_r)^T$, **therefore**

$$\frac{\partial\Phi}{\partial x_1} = 2\sum_{i=1}^{n}(x_1 - f_{ii})$$

and

$$\frac{\partial\Phi}{\partial x_s} = 2\{\sum_{i=1}^{n-s}(x_{s+1} - f_{i+s,i}) \ + \ (x_{s+1} - f_{i,i+s})\}$$

where $s = 1, \cdots, n-1$. **Differentiating again gives**

$$\frac{\partial^2\Phi}{\partial x_r\partial x_s} = 0 \qquad \textit{if} \quad r \neq s,$$

$$\frac{\partial^2\Phi}{\partial x_1^2} = 2(n)$$

and

$$\frac{\partial^2\Phi}{\partial x_{s+1}^2} = 4(n-s) \tag{14}$$

where $s, r = 1, \cdots, n-1$**.**

The simple form of (8) is utilized by writing the constraints $D_2(D) = 0$ **in the following form**

$$\begin{aligned} d_{ii}(\mathbf{x}) &= x_1 - \sum_{k,l=1}^{r} x_{i-k+1}[A_{11}^{-1}]_{kl}\ x_{i-l+1} = 0 \\ d_{ij}(\mathbf{x}) &= x_{|i-l+1|} - \sum_{k,l=1}^{r} x_{|i-k+1|}[A_{11}^{-1}]_{kl}\ x_{|i-l+1|} = 0 \end{aligned}$$

where $i, j = m+1, \cdots, n$ **and** $[A_{11}^{-1}]_{st}$ **means the element of** A_{11}^{-1} **in** st **position.**

Thus (10) expressed as

$$\begin{aligned} &\text{minimize} \quad \Phi = \sum_{i,j=1}^{n} (f_{ij} - x_{|i-j+1|})^2. \\ &\text{subject to } d_{ij}(\mathbf{x}) = 0 \end{aligned} \tag{15}$$

In order to write down the SQP method applied to (15) it is necessay to derive first and second derivatives of d_{ij} **which enables a second order rate of convergence to be achieved. Now**

Differentiating $A_{11}A_{11}^{-1} = I$ **gives**

$$\frac{\partial A_{11}}{\partial x_s}A_{11}^{-1} + A_{11}\frac{\partial A_{11}^{-1}}{\partial x_s} = 0 \qquad s = 1, \ldots, n-1$$

$$\Rightarrow \qquad A_{11}\frac{\partial A_{11}^{-1}}{\partial x_s} = -\frac{\partial A_{11}}{\partial x_s}A_{11}^{-1}$$

then

$$\frac{\partial A_{11}^{-1}}{\partial x_s} = - A_{11}^{-1}\frac{\partial A_{11}}{\partial x_s}A_{11}^{-1},$$

but since

$$\frac{\partial A_{11}}{\partial x_s} = I_s$$

where I_s **is** $m \times m$ **matrix given by**

$$I_s = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

where the "1" appearing in the first row is in the sth column and the "1" appearing in the first column is in the sth row. Hence the matrix I_s is a matrix that contains "1"s in two off diagonal and zeros elsewhere.

$$\frac{\partial A_{11}^{-1}}{\partial x_s} = - \ A_{11}^{-1} \ I_s \ A_{11}^{-1}. \tag{16}$$

Hence from (8)

$$\frac{\partial D_2}{\partial x_s} = \frac{\partial}{\partial x_s}(A_{22} - A_{21}A_{11}^{-1}A_{21}^T)$$

$$= II_s - III_sA_{11}^{-1}A_{21}^T + A_{21}A_{11}^{-1}I_sA_{11}^{-1}A_{21}^T - A_{21}A_{11}^{-1}III_s^T$$

where

$$\frac{\partial A_{22}}{\partial x_s} = II_s$$

and

$$\frac{\partial A_{21}}{\partial x_s} = III_s$$

matrices similar to I_s **with** II_s $n-m \times n-m$ **matrix contains ones in two off diagonal and zeros elsewhere and** III_s $n-m \times m$ **matrix contains ones in one off diagonal and zeros elsewhere.**

Let

$$V^T = -A_{21}^TA_{11}^{-1} \quad and \quad W = III_sV$$

then (17) become

$$\frac{\partial D_2}{\partial x_s} \;=\; II_s + V^T I_s V + W^T + W$$

Furthermore differentiating (16)

$$\frac{\partial^2 D_2}{\partial x_s \partial x_r} = Y + Y^T$$

where

$$Y = -Z_r^T A_{11}^{-1} Z_s \qquad and \qquad Z_t = I_t V - III_t^T$$

Therefore

$$\frac{\partial^2 d_{ij}}{\partial x_s \partial x_r} \;=\; y_{ij} + y_{ji}$$

where $i, j = m+1, \cdots, n.$

Now let

$$\begin{aligned} W &= \nabla^2 \mathcal{L}(\mathbf{x}, \Lambda) \\ &= \nabla^2 \Phi \;-\; \sum_{i,j=m+1}^{n} \lambda_{ij} \nabla^2 d_{ij} \end{aligned} \tag{17}$$

where $\nabla^2\Phi$ given by (14) and

$$\sum_{i,j=m+1}^{n} \lambda_{ij}\nabla^2 d_{ij} = \begin{bmatrix} \Sigma_{i,j}\lambda_{ij}\frac{\partial^2 d_{ij}}{\partial x_1 \partial x_1} & \cdots & \Sigma_{i,j}\lambda_{ij}\frac{\partial^2 d_{ij}}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \Sigma_{i,j}\lambda_{ij}\frac{\partial^2 d_{ij}}{\partial x_n \partial x_1} & \cdots & \Sigma_{i,j}\lambda_{ij}\frac{\partial^2 d_{ij}}{\partial x_n \partial x_n} \end{bmatrix}$$

Therefore the SQP method applied to (15) requires the solution of the QP subproblem

$$\begin{array}{l} \underset{\boldsymbol{\delta}}{\text{minimize}}\ \ \Phi + \nabla\Phi^T\boldsymbol{\delta} + \frac{1}{2}\boldsymbol{\delta}^T W\boldsymbol{\delta} \quad \boldsymbol{\delta} \ \in\ \mathbb{R}^m \\ \text{subject to } d_{ij} + \nabla d_{ij}^T\boldsymbol{\delta} = 0 \ \ i,j = m+1,\ldots,n \end{array} \quad (18)$$

giving a correction vector $\boldsymbol{\delta}^{(k)}$, so that $\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \boldsymbol{\delta}^{(k)}$. Also the Lagrange multipliers of the equations in (18) become the elements $\lambda_{ij}^{(k+1)}$ for the next iteration. Usually $\nabla^2\mathcal{L}$ is positive definite in which case, if $\mathbf{x}^{(k)}$ is sufficiently close to $\mathbf{x}^*$, the basic SQP method converges and the rate is second order (e.g. Fletcher [8])

An algorithm with better convergence properties is suggested by Fletcher [7] in which a different subproblem to (18) is solved expressed as

$$\begin{aligned}&\underset{\boldsymbol{\delta}}{\text{minimize}}\;\; \Phi + \nabla\Phi^T\boldsymbol{\delta} + \tfrac{1}{2}\boldsymbol{\delta}^T W\boldsymbol{\delta} + \sigma\,\Sigma \mid d_{ij} + \nabla d_{ij}^T\boldsymbol{\delta}\mid \\ &\text{subject to} \|\boldsymbol{\delta}\| \leq \rho \end{aligned} \tag{19}$$

The solution $\boldsymbol{\delta}^{(k)}$ of this problem is used in the same way as with (18).

Conclusions

In this paper we have studied certain problems involving the positive semi-definite matrix constraint, with the involving l_1SQP method. Also some Numerical works needs to be done.

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References

[1] S. Al-Homidan and R. Fletcher, in *Recent Advances in Nonsmooth Optimization,* Eds. D. Du, L. Qi and R. Womersley, (World Scientific Publishing Co. Pte. Ltd., Singapore, 1995).

[2] S. Al-Homidan, *J. Comp. App. Math.,* To appear (1998).

[3] Al-Homidan, S. *Proc. of NMA'98: 4th International Conference on Numerical Methods and Applications* (1998).

[4] J. P. Burg, D. G. Luenberger and D. L. Wenger, *Proc IEEE* 70(1982). Annals of Math. Studies No. **22**, Princeton Univ. Press.(1950).

[5] S. Cabay and R. Meleshko *SIAM J. Matrix Anal. Appl.* **14** (1993).

[6] G. Cybenko, *Circuits Systems Signal process* **1** (1982).

[7] R. Fletcher, *SIAM J. Control and Optimization* **23**, 493(1985).

[8] R. Fletcher, *Practical methods of Optimization,* Wiley (1987).

[9] S. Y. Kung, *Theory of Networks and Systems* **Vol IV** (1981).