Chapter 8

Conclusions and further work

In this thesis we have studied certain problems involving positive semi–definite matrix constraint. We have found that our implementations of the new unconstrained methods for solving the Euclidean distance matrix problem have performed well in comparison with the projection method. However, the hybrid methods in Chapter 4 performed even better, with very fast convergence, especially the projection–unconstrained method (Section 4.3) which is much better than the projection method and Method 3.4.2 from which it is composed. In determining the correct rank the projection method worked well and found the rank in a few iterations. Also we have successfully found methods for switching from one method to another.

A number of suggestions for further research about the methods that solves the Euclidean distance matrix problem are the following.

- It is clear that if the diagonal matrix $\Delta^{(k)}$ in (3.3.11) satisfies $P_d P_M(F + \Delta^{(k)}) = P_M(F + \Delta^{(k)})$ then $P_M(F + \Delta^{(k)})$ is the required solution where F is a given matrix. Possibly from the structure of the given matrix F that one can find the required diagonal matrix in one go. It is not clear how to do this but it might be worth trying.

- The unconstrained methods have a large number of variables $(\sim (r-1)n$ depending on the method) which means that the method takes a large number of line searches to solve the problem. Therefore it is worth trying to restate the problem with only the diagonal matrix $\Delta^{(k)}$ as variables and then finding methods for solving it.

- Method 3.4.3 needs more investigation because the number of variables is less than the other unconstrained methods, whilst the number of line searches is larger. However number of possible reasons have been given in Section 3.6.

For the least distance problem in Chapter 5 two methods are developed, that is the projection method and the l_1 SQP method. The l_1 SQP method has performed well in comparison with the projection method which takes a huge number of iterations to solve the problem. Also the projection– l_1 SQP method (Section 4.3) has worked well in solving the problem. The few iterations taken by the projection method to determine the rank saves a large number of iterations taken by the l_1 SQP method. The integer s in Algorithm 5.4.1 chosen to be small $(\sim\ 2)$ in Table 5.5.2. This reduces the number of iterations taken by projection method although the rank is not accurately estimated and the lower bound given by Fletcher [1985] has worked better in the case 1–18.

Two suggestions for further research about the methods that solves the least distance problem are given in the following.

- By looking at problem (5.4.3), there is a different problems with every different initial vector $\mathbf{a}$. The projection Algorithm 5.2.2 solves this problem with the initial vector zero replaced by $\mathbf{a}$. Extending the l_1 SQP method to solve problems of this type is worth investigation.

- A modified projection algorithm similar to Algorithm 4.2.1 is needed for the least distance problem this enable us to use the result matrix from the l_1 SQP method as an initial matrix for the projection method. Then a more effective hybrid method could be obtained.

Two methods have given for solving the educational testing problem. One is the l_1 SQP method by Fletcher [1985] the other is the projection method by Glunt [1991]. The hybrid methods developed in Chapter 7 have good rate of convergence specially the l_1 SQP–projection method (Section 7.3) as compared with the methods of Chapter 6. The projection method is not very effective in determining the rank when $n \geq 12$. This is because a small value of s is shosen in Algorithms 7.2.1 and 7.3.1. In the other hand if s is increased then a large number of iterations are consumed by the projection method. Hence a suitable way of chosing the integer s is needs some investigation. Various examples are solved in Sections 6.5 and 7.4 with different τ. The best way to choose τ is given there.

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