Chapter 6

Algorithms for solving the educational testing problem

6.1 Introduction

The problem to be considered here is the educational testing problem. Such optimization problems come up in many practical situations, particularly in statistics where we have a matrix F which is usually a covariance matrix with varying elements. The educational testing problem is; given a symmetric positive definite matrix F how much can be subtracted from the diagonal of F and still retain a positive semi–definite matrix this can be expressed as

$$\begin{aligned} &\mathit{maximize} \quad \mathbf{e}^T\boldsymbol{\theta} \qquad \boldsymbol{\theta} \in \Re^n \\ &\mathit{subject\ to} \quad F - \mathit{diag}\,\boldsymbol{\theta} \geq 0 \\ &\qquad\qquad \theta_i \geq 0 \quad i = 1, ..., n \end{aligned} \tag{6.1.1}$$

where $\mathbf{e} = (1, 1, ..., 1)^T$. An equivalent form to problem (6.1.1) is

$$\begin{aligned} &\mathit{minimize} \quad \mathbf{e}^T\mathbf{x} \qquad \mathbf{x} \in \Re^n \\ &\mathit{subject\ to} \quad \bar{F} + \mathit{diag}\,\mathbf{x} \geq 0 \\ &\qquad\qquad x_i \leq v_i \quad i = 1, ..., n \end{aligned} \tag{6.1.2}$$

where $\bar{F} = F - Diag\, F$, and $diag\, \mathbf{v} = Diag\, F$.

An early approach in solving the educational testing problem is due to Bentler [1972]. He writes $F - diag\, \boldsymbol{\theta} = CC^T$, where C is unknown and minimizes trace (CC^T) subject to certain conditions. He found that there are a large number of variables, and also it does not account for the bounds $\theta_i \geq 0 \quad \forall\, i$. Furthermore, some difficulties in convergence to the optimum solution arise.

Woodhouse and Jackson [1977] have given a method for solving the problem by searching in the space of $\boldsymbol{\theta}$. However their method does not work efficiently and failed for particular examples.

Fletcher [1981b] has solved the problem in which the semi–definite constraint is reduced to an eigenvalue constraint and standard nonlinear programming techniques are used. But still some difficulties arise with the rates of convergence. Also the presumption that the eigenvalue constraint would be smooth at the solution, except in rare cases, is not correct and in fact the majority of such problems are nonsmooth at the solution.

In 1985 Fletcher developed a different algorithm for solving the educational testing problem. He gives various iterative methods for solving the nonlinear programming problem derived from the educational testing problem (6.1.2) using sequential quadratic programming techniques. One of these algorithms is the use of an l_1 exact penalty function. This algorithm works well with second order convergence and the function converging to the optimal solution. The only problem in these algorithms is the requirement to know the exact rank for the matrix $A^* = \bar{F} + diag\, \mathbf{x}^*$ where $\mathbf{x}^*$ solves (6.1.2).

Finally, Glunt [1991] describes a projection method for solving the educational testing problem. His idea is to construct a hyperplane L_τ in $\Re^n$ and then carry out the method of alternating projections (the von Neumann Algorithm 2.2.2) between the convex set $K = K_\Re \cap K_{off} \cap K_b$ and the hyperplane L_τ. His method converges globally and the order of convergence is very slow.

The statistical background involved in the educational testing problem is described in Section 6.2. In Section 6.3 the educational testing problem is solved using the theory developed in Section 2.3. Section 6.4 contains a brief description of the l_1 SQP method for solving problem (6.1.2). Finally in Section 6.5 numerical comparisons of these methods are carried out.

In Chapter 7 hybrid methods are considered. The projection method converges linearly or slower while the l_1 SQP method converges at second order but it requires the correct rank

r^* which can be gained from the projection method. Therefore, hybrid methods that take the advantage of both projection and l_1 SQP methods are described in Chapter 7.

6.2 The educational testing problem

This section explains the educational testing problem which arises from statistics. The problem is to find lower bounds for the reliability of the total score on a test (or subtests) whose items are not parallel using data from a single test administration. The educational testing problem consists of a number of student (N) taking a test or examination consisting of (n) subtests. The problem is to find how reliable is the students's total score in the sense of being able to

reproduce the same total on two independent occasions. Specifically it is required to know what evidence about reliability can be obtained by carrying out a test on one occasion only.

In this thesis we do not develop the entire theory (see Fletcher [1981b]) but just give enough information to construct the test problem (6.1.1). The data for the problem is an $N \times n$ table of scores $[X_{ij}]$ (see Table 6.2.1) such that X_{ij} gives the observed score of student i on subject j.

Define the mean observed score of subject j by

$$\bar{X}_j = \frac{1}{N} \sum_i X_{ij}. \tag{6.2.1}$$

Then the $n \times n$ matrix F given in (6.1.1) is constructed from an $N \times n$ data matrix $[X_{ij}]$ in the following way

$$f_{jk} = \frac{1}{N - 1} \sum_i (X_{ij} - \bar{X}_j)(X_{ik} - \bar{X}_k) \tag{6.2.2}$$

see Guttman [1945]. Then problem (6.1.1) is constructed with $\boldsymbol{\theta}$ as the unknown vector. For more about the statistical background for the educational testing problem and references see Fletcher [1981b].

15	25	20	28	35	50	21	18	22	28	28	12	15	40	18	23	14	16	15	10
21	27	32	32	41	42	30	35	33	32	64	16	24	38	34	13	15	17	28	18
23	35	40	22	55	48	36	40	46	18	38	18	26	37	24	24	17	20	19	26
23	29	50	36	42	52	44	32	24	19	32	24	20	46	32	23	11	12	40	28
34	37	42	19	36	46	17	26	35	28	39	54	21	47	29	42	18	18	30	20
36	60	70	45	55	54	32	30	32	29	41	28	20	47	36	28	20	20	18	24
36	35	46	27	50	40	60	34	39	46	48	63	20	48	40	19	21	24	40	22
38	70	44	50	45	42	20	28	29	16	55	40	22	49	42	25	23	26	30	28
39	46	52	24	37	60	53	30	46	43	54	54	23	46	44	22	35	22	48	30
40	74	65	60	72	41	33	36	24	52	64	36	28	50	46	26	25	23	30	30
40	48	32	23	58	52	23	40	37	24	58	38	29	54	44	28	11	27	41	34
41	12	24	50	47	48	41	42	37	28	56	57	32	51	43	25	17	24	32	39
46	52	76	48	70	58	20	50	28	42	76	58	28	58	45	34	27	35	18	56
46	73	84	63	38	57	33	56	42	18	72	77	31	52	48	32	35	32	32	19
47	42	74	28	60	57	36	42	48	52	63	46	36	53	49	53	29	30	42	40
47	82	72	70	39	64	21	25	44	26	44	44	37	51	46	33	38	35	37	42
47	40	42	50	48	61	40	40	26	29	61	44	30	56	47	52	46	37	48	40
48	70	65	48	42	57	35	58	50	46	60	32	34	58	54	35	36	31	16	18
49	65	60	55	62	56	52	50	52	28	50	48	34	58	53	41	45	40	38	52
50	30	35	28	62	54	41	46	50	21	65	33	32	58	54	38	50	44	43	50
52	42	54	33	42	64	40	40	56	44	64	38	34	60	44	34	38	30	44	38
52	72	70	65	72	68	62	38	56	44	58	46	36	58	46	36	55	20	48	47
52	44	64	72	44	62	35	44	56	46	62	39	30	61	46	38	40	42	24	80
53	25	42	28	68	52	41	45	44	26	28	43	51	62	47	35	42	48	50	40
54	48	60	58	36	51	63	41	64	29	63	49	32	58	47	39	43	58	48	49
55	64	62	30	42	57	34	47	52	34	57	37	43	63	48	38	47	20	54	65
58	30	24	62	51	51	44	36	43	25	36	54	41	65	48	43	40	35	50	42
58	16	40	45	42	58	44	42	58	36	58	52	40	64	49	36	18	45	53	28
58	44	56	51	68	68	46	48	72	38	62	34	32	68	49	42	47	47	28	70
59	58	58	50	74	52	36	58	60	28	44	56	34	72	51	33	48	58	54	58
60	32	35	48	40	56	52	32	40	37	72	57	36	61	52	51	42	46	17	51
60	78	80	62	52	54	58	47	80	32	64	39	45	66	53	42	70	40	50	18
60	38	55	66	42	52	30	54	62	42	90	38	38	63	56	46	62	48	55	44
61	48	64	68	70	53	42	40	38	45	73	56	50	64	54	46	45	42	50	22
62	86	94	50	49	62	48	56	74	33	84	36	52	67	52	47	41	70	57	53
63	35	38	55	38	58	46	59	63	48	62	58	38	68	53	47	48	75	44	25
64	79	65	76	68	57	32	33	52	46	72	62	52	54	54	48	55	35	60	54
64	50	52	35	60	56	52	64	76	36	63	44	48	56	52	49	63	38	48	46

Table 6.2.1:(to be continued in the next page)

65	37	42	70	50	58	58	62	66	34	53	64	62	72	53	50	74	45	62	57
65	82	74	63	36	62	60	58	60	38	57	63	58	70	51	51	55	55	64	58
67	44	46	54	52	58	55	68	80	40	28	65	50	71	54	52	65	70	74	68
67	48	56	80	44	64	62	35	70	62	58	72	56	72	60	41	78	38	75	68
68	62	78	56	50	53	62	54	80	64	62	48	54	74	62	76	58	45	80	36
69	39	30	42	38	62	40	32	68	56	68	71	58	73	56	43	79	47	73	70
70	52	20	76	69	61	64	56	69	54	64	66	58	78	52	72	32	47	71	71
72	54	72	38	54	51	66	65	76	72	54	49	60	74	57	42	68	62	22	46
72	42	48	70	70	57	42	40	68	53	62	74	60	78	58	52	55	48	72	75
74	64	66	70	42	60	40	78	53	48	69	67	76	77	59	68	58	55	16	60
78	68	62	63	35	56	63	80	74	43	71	78	62	76	59	68	70	75	72	75
79	37	42	28	64	52	40	38	72	72	56	52	58	82	60	61	75	66	58	58
80	62	30	65	59	51	68	57	74	48	78	71	42	54	61	62	78	69	78	58
82	85	80	52	44	57	70	69	64	71	85	76	64	56	63	67	44	70	60	26
82	40	74	52	52	64	74	64	76	46	64	46	51	80	63	68	65	55	70	67
84	42	76	70	55	61	90	80	56	41	58	82	72	72	67	73	85	60	76	78
84	42	54	60	42	58	42	60	52	70	77	68	68	74	59	71	79	65	44	76
86	85	88	80	37	63	80	72	79	42	73	42	68	82	65	73	85	62	80	82
87	53	51	62	68	56	60	42	78	42	62	74	62	84	65	75	63	75	68	83
89	41	60	40	60	54	88	88	83	57	84	64	64	80	62	65	90	78	88	52
90	73	78	77	52	42	56	50	58	59	72	84	70	84	62	63	82	85	78	87
90	81	74	64	48	38	86	52	80	63	66	68	60	62	63	74	75	81	84	94
96	85	88	90	72	44	58	62	70	74	64	74	72	64	64	84	85	78	88	54
97	56	55	35	68	70	78	76	56	72	83	69	65	86	65	76	82	89	92	90
99	75	65	88	54	42	80	90	88	58	78	88	70	88	63	82	72	71	98	80
100	65	75	70	70	60	83	85	70	62	72	90	72	84	64	78	88	80	80	72

Table 6.2.1: The Woodhouse [1976] data which corresponds to 64 students and 20 subtests.

6.3 A projection algorithm for solving the educational testing problem

In this section a projection algorithm for solving the educational testing problem is described. The method described here depends on Algorithm 2.3.1 developed in Section 2.3.

The constraints in problem (6.1.2) can be expressed as

$$\bar{F} + diag\,\mathbf{x} \in K_{\Re} \cap K_{off} \cap K_b.$$

Then problem (6.1.2) can be expressed as

$$\begin{aligned} &\underset{\mathbf{x}}{\text{minimize}} \quad \mathbf{e}^T\mathbf{x} \qquad \mathbf{x} \in \Re^n \\ &subject\ \ to\ \ \bar{F} + diag\,\mathbf{x} \in K_{\Re} \cap K_{off} \cap K_b \end{aligned} \tag{6.3.1}$$

where $K_{\Re}$, K_{off} and K_b are given in (1.3.1), (1.3.5) and (1.3.6) respectively. Therefore problem (6.3.1) is a special case of problem (2.1.4) which can be solved by Algorithm 2.3.1. To solve (6.3.1) in this way we need the hyperplane L_τ given by (2.3.1) and we define $K = \bigcap_{i=1}^{m} K_i$ by $K = K_{\Re} \cap K_{off} \cap K_b$. However L_τ must be defined on the space of $n \times n$ matrices, and this can be done by

$$\begin{aligned} L_\tau &= \{Y = \bar{Y} + diag\, \mathbf{y} \in \Re^{n\times n} |\ \mathbf{e}^T \mathbf{y} = \tau\} \\ &= \{Y \in \Re^{n\times n} |\ tr(Y) = \tau\} \end{aligned} \tag{6.3.2}$$

where $Diag\, Y = diag\, \mathbf{y}$ and τ is chosen such that

$$\tau < \min_{\mathbf{x}\in K} \mathbf{e}^T \mathbf{x} \tag{6.3.3}$$

We also need the projection $P_{L_\tau}(Y)$, and following (2.3.8) with $\mathbf{e}$ replaced by I we can write

$$P_{L_\tau}(Y) = Y + \frac{\tau - tr(Y)}{n} I. \tag{6.3.4}$$

In Algorithm 2.3.1 the projection $P_K(.)$ in (2.3.4) is given. In particular for problem (6.1.2) we need the projection $P_K(A)$ where $K = K_{\Re} \cap K_{off} \cap K_b$ for any matrix A. In fact this projection was solved by Algorithm 5.2.2 and hence we just include Algorithm 5.2.2 as an inner iteration inside the following algorithm which is a special case of Algorithm 2.3.1. This algorithm solves the educational testing problem.

Algorithm 6.3.1

Given any positive definite matrix F, let $F^{(0)} = F$

$$\begin{aligned} &For \quad k = 1,\ 2,\ \ldots \\ &\qquad F^{(k+1)} = P_{L_\tau}(F^{(k)}) \\ &\qquad For \quad l = 1,\ 2,\ \ldots \end{aligned} \tag{6.3.5}$$

$$
\begin{aligned}
&\qquad A^{(0)} \;=\; F^{(k+1)} \\
&\qquad A^{(l+1)} \;=\; A^{(0)} \;+ P_b P_{off} P_{\Re}(A^{(0)}) \;-\; P_{\Re}(A^{(0)}) \\
&End \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (6.3.6) \\
&F^{(k+1)} \;=\; F^{(0)} \;-\; diag\; F^{(0)} + diag\; P_{\Re}(A^*)
\end{aligned}
$$

End

where A^* is the solution for the inner iteration and $P_{\Re}$, P_{off} and P_b are given in (5.2.3), (5.2.5) and (5.2.6) respectively.

From Theorem 2.3.2 $P_{\Re}(A^{(k)})$ and $P_b P_{off} P_{\Re}(A^{(k)})$ converges to the solution of problem (6.3.1). Also in Theorem 2.3.2 $\mathbf{x}_1^{(k)} \equiv F^{(k)}$ and $\mathbf{x}_2^{(k)} \equiv A^{(k)}$. Equations (6.3.5)–(6.3.6) are the inner loop and they are the same as Algorithm 5.2.2. In Section 6.5 numerical results for Algorithm 6.3.1 are given.

6.4 The l_1SQP method

This section contains a brief description of l_1SQP method for solving the educational testing problem. The l_1SQP methods in Section 1.7 are used. This method was given by Fletcher [1985].

The constraints in problem (6.1.2) can be expressed as

$$\bar{F} \;+\; diag\,\mathbf{x} \;\in\; K_{\Re} \;\cap\; K_{off} \;\cap\; K_b.$$

Then problem (6.1.2) can be expressed as

$$
\begin{aligned}
&\underset{\mathbf{x}}{\text{minimize}} \quad \mathbf{e}^T\mathbf{x} \qquad \mathbf{x} \;\in\; \Re^n \\
&subject\;\; to\;\; \bar{A} \;+\; diag\,\mathbf{x} \;\in\; K_{\Re} \;\cap K_{off}, \;\; \mathbf{x} \;\leq\; \mathbf{v}
\end{aligned}
\qquad (6.4.1)
$$

where $diag\,\mathbf{v} \;=\; Diag\; A^{(0)}$. This problem is similar to problem (5.3.1) in Section 5.3, therefore the method is given in detail in that section. Thus for solving problem (6.4.1) one follows the details of Section 5.3, changing only the definition of the objective function. However in this section we give a summary of what has been given in Section 5.3. The first order necessary conditions for $\mathbf{x}^*$ to solve (6.4.1) are similar to what given in (5.3.4) with the condition (5.3.4a) replaced by

$$\mathbf{e} \ + \ \mathbf{b}^* \ + \ \boldsymbol{\pi}^* \ = \ 0.$$

It is difficult to deal with the matrix cone constraints in (6.4.1), since it is not easy to specify if the elements are feasible or not. An equivalent problem to (6.4.1) with the constraint $D_2 \ = \ \mathbf{0}$ is considered. This problem is similar to problem (5.3.5) with the objective function $\mathbf{x}^T\mathbf{x}$ replaced by $\mathbf{e}^T\mathbf{x}$. This formulation will enable us to derive algorithms with a second order rate of convergence.

Now using the constraint $D_2 \ = \ \mathbf{0}$ in the form (5.3.9), this will produce an equivalent problem to (6.4.1). The number of variables in this new problem can be reduced to r variables which gives the new reduced problem

$$\begin{aligned} &\underset{\mathbf{x}}{\text{minimize}} \quad f(\mathbf{x}) \ = \ \sum_{k=1}^{r} x_k \ + \ \sum_{i=r+1}^{n} x_i(\mathbf{x}) \\ &\textit{subject to} \ \ d_{ij}(\mathbf{x}) \ = \ 0, \quad i \ \neq \ j, \quad \mathbf{x} \ \leq \ \mathbf{v}. \qquad i,j \ = \ r+1,\ldots,n \end{aligned} \tag{6.4.2}$$

The expressions for the derivatives $\frac{\partial d_{ij}}{\partial x_s}$ and $\frac{\partial^2 d_{ij}}{\partial x_s \partial x_t}$ given in (5.3.13) and (5.3.14) respectively enable us to finding expressions for ∇f, $\nabla^2 f$ and $W^{(k)}$. Then using these expressions the QP subproblem

$$\begin{aligned} &\underset{\boldsymbol{\delta}}{\text{minimize}} \quad f^{(k)} \ + \ \nabla f^{(k)}\boldsymbol{\delta} \ + \ \tfrac{1}{2}\,\boldsymbol{\delta}^T W^{(k)}\boldsymbol{\delta} \qquad\qquad \boldsymbol{\delta} \ \in \ \Re^r \\ &\textit{subject to} \ \ d_{ij}^{(k)} \ + \ \nabla\, d_{ij}^{(k)T}\boldsymbol{\delta} \ = \ 0 \qquad i \ \neq \ j \quad i,j = r+1,\ldots,n \\ &\qquad\qquad\qquad \mathbf{x}^{(k)} \ + \ \boldsymbol{\delta} \ \leq \ \mathbf{v} \end{aligned} \tag{6.4.3}$$

is defined. Thus the SQP method applied to (6.4.2) requires the solution of the QP subproblem (6.4.3). The matrix $W^{(k)}$ is positive semi–definite see Fletcher [1985].

It is shown in Section 1.7 that the SQP method may not converge globally and it is usually modified by the exact penalty function (5.3.25). Now a similar technique to what stated in Section 5.3 is followed to take over the problem of non–globality convergent using the l_1 exact penalty function (5.3.25).

This section is concluded by some restrictions and conditions in similar manner to those considered at the end of Section 5.3.

For more about the l_1 SQP methods for solving the educational testing problems see Fletcher [1985].

6.5 Numerical results and comparisons

In this section numerical problems are obtained from the data given in Table 6.2.1, by Woodhouse [1976]. The Woodhouse data set is a 64×20 data which corresponds to 64 students and 20 subtests. Various selections from the set of subsets of columns are used to give various test problems to form the matrix A. These subsets are those given in the first columns of Tables 6.5.1–2, the value of n is the number of elements in each subset. Equation (6.2.2) gives the formula for calculating the educational testing problems from Table 6.2.1.

In Algorithm 6.3.1 τ must satisfy the condition (6.3.3). Since $\mathbf{x}^*$ not known in advance and with elements $f_{ij} \overset{\approx}{>} 100$ then it is clear that the diagonal elements $\bar{F} + diag\,\mathbf{x}^{(k)}$ is greater than about 100 so $\mathbf{e}^T\mathbf{x} \overset{\approx}{>} 100n$ since F is positive definite. Therefore from (6.3.3) the choice $\tau = 100$ is recommended. In fact we recommend this choice since the elements f_{ij} are close to each either in magnitude. However, in general the off-diagonal elements can play a role in making a better estimate for τ. If τ chosen randomly and does not satisfy the condition (6.3.3) then the matrix $F - diag\,\mathbf{x}^{(k)}$ is not positive semi–definite and the method is rerun with different τ. In Chapter 7 more information is available and a different strategy is followed.

Glunt [1991] and Fletcher [1985] tested their methods on the twelve test problems originally due to Woodhouse [1976]. The same test problems are applied for the methods in this chapter. This section contains numerical results for the projection method given in Table 6.5.1. Numerical results for the l_1 SQP algorithm are given in Table 6.5.2. In all the tables of this section NOI gives the number of outer iteration when solved by Algorithm 6.3.1, TNII gives the total number of inner iteration used by Algorithm 5.2.2 in Algorithm 6.3.1 and $r^{(0)}$ gives the number of positive eigenvalues in the first iteration of Algorithm 6.3.1.

In Table 6.5.1 a comparison between $\tau = -100$ and $\tau = 100$ is given for the same test problems using Algorithm 6.3.1. We choose $\tau = -100$ for comparison purposes which shows that when τ is remote from condition (6.3.3) then the method takes more inner iterations. It is clear that with $\tau = 100$ the method takes fewer inner iterations in most of the examples. Because of Algorithm 5.2.2 the projection method is very slow and the number of iterations taken by the projection method is very large especially when the bounds are active. The results obtained by the l_1SQP method of Section 6.4 are tabulated in Table 6.5.2 as given by Fletcher [1985] and mentioned here for comparison purposes. The iterates converge to essentially the same values of $\mathbf{x}^*$ in both methods.

The projection method is very expensive in the sense that it consumed a large number of iterations whilst the l_1SQP method takes a very small number of iterations.

The NAG routine is used here to find the eigenvalues and eigenvectors for the matrix $\bar{F} + diag\ \mathbf{x}^{(k)}$. This matrix is reduced to a real symmetric tridiagonal matrix by Householder's method. Then the eigenvalues and eigenvectors are calculated using the QL algorithm. The amount of work required by these algorithms is approximately $\frac{4}{3}n^3$ multiplications per one inner iteration (Golub and Van Loan [1989]).

Again the NAG routine is used this time for solving the QP subproblem (6.4.3) which is one iteration of the SQP method. The method used by the NAG routine to solve the QP subproblem requires the solution for the system

$$Z^{(k)}\ W\ Z^{(k)T}\mathbf{p}^{(k)} \ = \ -\,Z^{(k)T}(\mathbf{c} \ + \ W\ \mathbf{x}^{(k)}) \tag{6.5.1}$$

where $\mathbf{c} = \nabla f$ and $Z^{(k)}$ is a matrix whose columns form a basis for the null space of $A^{(k)}$ (the matrix of coefficients of the bounds and active constraints). $\mathbf{p}^{(k)}$ is a search direction. The matrix $Z^{(k)}$ is obtained from the TQ factorization of $A^{(k)}$, in which $A^{(k)}$ is represented as

$$A^{(k)} \begin{bmatrix} Z^{(k)} \\ Q \end{bmatrix} \ = \ [\,\mathbf{0} \quad T^{(k)}\,]. \tag{6.5.2}$$

The Lagrange multipliers $\boldsymbol{\lambda}^{(k)}$ are defined as the solution of the system

$$A^{(k)}\ \boldsymbol{\lambda}^{(k)} \ = \ \mathbf{c} \ + \ W\ \mathbf{x}^{(k)}. \tag{6.5.3}$$

Eqautions (6.5.1) and (6.5.2) costs approximately $\frac{7}{3}n^3$ multiplications to solve and (6.5.3) costs approximately $\frac{8}{3}n^3$ multiplications to solve (Golub and Van Loan [1989]). Thus one iteration of the SQP method costs approximately $\frac{15}{3}n^3$ multiplications.

Thus one iteration of the SQP method costs about 6 times greater as one iteration of the projection method. Nonetheless the SQP method is much better than the projection method since the number of iterations taken by the projection method is about 60 times greater than the number of iterations taken by the SQP method. However in Chapter 7 hybrid methods are carried out which use even fewer iterations.

Columns which determine A	$\tau \;=\; -100$		$\tau \;=\; 100$	
	NOI	TNII	NOI	TNII
1,2,5,6	3	197	4	240
1,3,4,5	2	224	3	266
1,2,3,6,8,10	3	580	3	522
1,2,4,5,6,8	4	4994	4	4518
1–6	3	1351	3	1243
1–8	4	1948	4	1702
1–10	3	2918	3	2534
1–12	3	2403	3	2442
1–14	3	3196	3	3143
1–16	3	5215	3	4796
1–18	3	14043	3	14171
1–20	3	8255	3	7978

Table 6.5.1: Results for the educational testing problem from the projection Algorithm 6.3.1

Columns which determine A	$r^{(0)}$	r^*	NQP	$\sum \theta_i^*$
1,2,5,6	2	3	14	542.77356
1,3,4,5	2	2	12	633.15784
1,2,3,6,8,10	3	5	9	305.48170
1,2,4,5,6,8	3	4	13	564.46331
1–6	3	4	14	535.36227
1–8	5	6	29	641.83848
1–10	6	8	34	690.78040
1–12	8	9	29	747.48921
1–14	10	12	36	671.27506
1–16	11	14	42	663.46204
1–18	13	15	27	747.50574
1–20	15	18	39	820.34265

Table 6.5.2: Results for the educational testing problem from the l_1SQP method of Section 6.4.

Table 6.5.3 investigates the effect of varying τ. It shows the outcome from Algorithm 6.3.1 for the following example

$$\bar{F} = \begin{bmatrix} 0 & 1 & 2 & -2 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ -2 & 2 & 1 & 0 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 2 \\ 4 \\ 8 \\ 10 \end{bmatrix}$$

with different τ. From Table 6.5.3 it is clear that small τ increases the total number of iterations performed by Algorithm 5.2.2, whilst on the other hand bigger τ decreases the total number of inner iterations and increases the number of outer iterations which are very cheap to calculate using the projection (6.3.4) which costs approximately n multiplications while one inner iteration costs approximately $\frac{4}{3}n^3$ multiplications. Hence it is recommended to increase τ to be close to the boundary of the condition (6.3.3) which is compatible with the choice in Table 6.5.1.

τ	NOI	TNII	$\sum x_i^*$	$r^{(0)}$	r^*
-30.0	2	2679	15	0	2
-20.0	2	2215	15	1	2
-10.0	2	1734	15	2	2
-5.0	2	1571	15	2	2
0.0	2	1291	15	2	2
5.0	3	1308	15	2	2
10.0	3	960	15	2	2
14.0	6	787	15	2	2
14.9	15	891	15	2	2
15.0	30	792	15.0051	2	2

Table 6.5.3: Numerical comparisons for same example with different τ.