Q >

$$LU=\begin{bmatrix}1&0&0\\-\frac{1}{3}&1&0\\\frac{1}{3}&\frac{1}{2}&1\end{bmatrix}\begin{bmatrix}9&-3&3\\0&4&2\\0&0&4\end{bmatrix}=\begin{bmatrix}9&0&0\\-3&4&0\\3&2&4\end{bmatrix}\begin{bmatrix}1&-\frac{1}{3}&\frac{1}{3}\\0&1&\frac{1}{2}\\0&0&1\end{bmatrix}$$

>

$Ax=b \Rightarrow LUx=b \quad Ly=b$ where $y=Ux$

$$\begin{bmatrix}9&0&0\\-3&4&0\\3&2&4\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}0\\0\\1\end{bmatrix} \Rightarrow y_1=0,\ y_2=0,\ y_3=\frac{1}{4}$$

then $\begin{bmatrix}1&-\frac{1}{3}&\frac{1}{3}\\0&1&\frac{1}{2}\\0&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\\frac{1}{4}\end{bmatrix}$ $\quad x_3=\frac{1}{4}$, $x_2=-\frac{1}{8}$, $x_1=-\frac{1}{8}$

$$> \quad LDL^t=\begin{bmatrix}1&0&0\\-\frac{1}{3}&1&0\\\frac{1}{3}&\frac{1}{2}&1\end{bmatrix}\begin{bmatrix}9&&0\\0&4&\\&&4\end{bmatrix}\begin{bmatrix}1&-\frac{1}{3}&\frac{1}{3}\\0&1&\frac{1}{2}\\0&0&1\end{bmatrix}$$

$$> \quad LL^t=\begin{bmatrix}3&0&0\\-1&2&0\\1&1&2\end{bmatrix}L^t$$

Q2.> $\lambda_1=1,\ \lambda_2=2 \quad \rho(\lambda)=\max\{1,2\}=2$

Q> $\|A\|_2=\sqrt{\rho(AA^T)}$ $\quad AA^T=\begin{bmatrix}4&4\\4&4\end{bmatrix}$ $\quad \lambda_1=\frac{9+\sqrt{65}}{2}$, $\lambda_2=\frac{9-\sqrt{65}}{2}$

$$=\sqrt{\frac{9+\sqrt{65}}{2}}$$

$$\|A\|_F=\sqrt{1+2^2+2^2}=\sqrt{1+4+4}=3$$

$$\|A\|_\infty=3$$

> $K(A)=\|A\|_\infty\|A^{-1}\|_\infty=3\cdot2=6$ the $K(A)$ is close to 1 relatively So we can see it is well conditioned.

> $x_i^{(k)} = (1-\omega) x_i^{(k-1)} + \frac{\omega}{a_{ii}} \left[b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k-1)} \right]$

$x_1^{(1)} = -1 \qquad x_2^{(1)} = 0.4$

$\|x\|_1 = \sum |x_i|$

$\|A\|_1 = \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}|$

$\|A\|_1 = 4 \qquad \|A^{-1}\|_1 = 1.5$

> $r^{(1)} = \begin{bmatrix} -1.2 \\ -0.2 \end{bmatrix}$

> $\frac{\|x - \tilde{x}\|}{\|x\|} \le 6 \, \frac{1.4}{2} = 4.2$

> $\sqrt{\frac{\alpha^2 + 4\alpha\beta + 8\beta^2}{\alpha^2 + \beta^2}}$

Q3 $r = b - A\tilde{x} = \begin{bmatrix} 3 \\ 3.0001 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 1.0001 & 2 \end{bmatrix} \begin{bmatrix} 1.1 \\ 0.9 \end{bmatrix}$

$= \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}$

then $A\tilde{y} = r \Rightarrow \begin{bmatrix} 1 & 2 \\ 1.0001 & 2 \end{bmatrix} \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix} = \begin{bmatrix} 0.1 \\ 0. \end{bmatrix}$

$\Rightarrow \tilde{y} = \begin{bmatrix} -0.1 \\ 0.1 \end{bmatrix} \Rightarrow x = \tilde{x} + \tilde{y} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Q4 $\|V\|_1 = 6, \quad \|V\|_2 = \sqrt{14}, \quad \|V\|_\infty = 3$

> S.D.D. $\Rightarrow |a_{ii}| > \sum_{j=1, j\ne i}^{n} |a_{ij}| \Rightarrow \frac{\sum |a_{ij}|}{a_{ii}} < 1$

$\Rightarrow \left[D^{-1}(L+U) \right]_i < 1 \qquad i = 1, \dots, n$

since $D^{-1} = \begin{bmatrix} \frac{1}{a_{11}} & & \\ & \ddots & \\ & & \frac{1}{a_{nn}} \end{bmatrix}$ and $[L+U]_i = \sum_{j=1, i\ne j}^{n} |a_{ij}|$

$\Rightarrow \max \left[D^{-1}(L+U) \right]_i < 1 \Rightarrow \|T_j\|_\infty < 1.$

$$K(AB) = \|AB\| \, \|(AB)^{-1}\| = \|AB\| \, \|B^{-1}A^{-1}\| \leq \|A\| \, \|B\| \, \|B^{-1}\| \, \|A^{-1}\| = K(A)K(B)$$

See the book.

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