Each question worth 10 points

King Fuhd University of Petroleum and Minerals
Department of Mathematical Sciences

Math 101-5,9
First Exam , Semester 041
Time:8:00-9:10 pm, Tuesday, October 12, 2004

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Name :--ID # :--

Q1. Use the definition of derivative to find $f'(x)$ where $f(x) = \sqrt[2]{x-1}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{x+h-1} - \sqrt{x-1}}{h} \cdot \frac{\sqrt{x+h-1} + \sqrt{x-1}}{\sqrt{x+h-1} + \sqrt{x-1}}$$

$$= \lim_{h\to 0} \frac{x+h-1-x+1}{h\left(\sqrt{x+h-1}+\sqrt{x-1}\right)}$$

$$= \lim_{h\to 0} \frac{1}{\left(\sqrt{x+h-1}+\sqrt{x-1}\right)} = \frac{1}{2\sqrt{x-1}}.$$

Q2. Find values for the constants k and h that will make the following function continuous

$$f(x) = \begin{cases} \dfrac{\sin^3 kx}{x^3} & x < 0 \\ 8 & x = 0 \\ h + x^3 & x > 0 \end{cases}$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\sin^3 kx}{x^3} = \lim_{x\to 0^-} k^3 \frac{\sin^3 kx}{(kx)^3} = k^3\left[\lim_{x\to 0^-} \frac{\sin kx}{kx}\right]^3$$

$$= k^3[1] = k^3$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} h + x^3 = h.$$

Now $f(x)$ continuous at $x=0$ if

$$h = 8$$

$$k^3 = 8 = h \Rightarrow k = 2$$

Q7. Find $\lim\limits_{x\to\infty} \frac{\cos x}{x}$.

$x \geq 0$

Using squeezing th$^{\circ}$. $-1 \leq \cos x \leq 1 \Rightarrow -\frac{1}{x} \leq \frac{\cos x}{x} \leq \frac{1}{x}$

Since $\lim\limits_{x\to\infty} \frac{-1}{x} = 0$ and $\lim\limits_{x\to\infty} \frac{1}{x} = 0$ ①

Therefore $\lim\limits_{x\to\infty} \frac{\cos x}{x} = 0$

Q8. Find $\lim\limits_{x\to\infty} \sqrt{x^4 - x^2} - x^2$.

$$\cdot \frac{\sqrt{x^4 - x^2} + x^2}{\sqrt{x^4 - x^2} + x^2}$$

$$= \lim_{x\to\infty} \frac{\cancel{x^4} - x^2 - \cancel{x^4}}{\sqrt{x^4 - x^2} + x^2} = \lim_{x\to\infty} \frac{-x^2/x^2}{\sqrt{\frac{x^4}{x^4} - \frac{x^2}{x^4}} + \frac{x^2}{x^2}}$$

$$= \frac{-1}{\sqrt{1-0} + 1} = -\frac{1}{2}$$

Q9. $\lim\limits_{x\to\infty} \frac{1 + x^3 - 2x^5}{x + x^7 + 3x^5} = \lim\limits_{x\to\infty} \frac{\frac{1}{x^7} + \frac{x^3}{x^7} - \frac{2x^5}{x^7}}{\frac{x}{x^7} + \frac{x^7}{x^7} + \frac{3x^5}{x^7}}$

$$= \frac{0 + 0 - 0}{0 + 1 + 0} = 0$$

Q5. Show that there exists some $c \in (-\pi/2, \quad \pi/2)$ such that $\frac{c}{2} = \sin c$.

$$\boxed{C = 2\sin C \qquad C - 2\sin C = 0}$$

Let $f(x) = x - 2\sin x$

$$f(-\tfrac{\pi}{2}) = -\frac{\pi}{2} - 2\sin(-\tfrac{\pi}{2}) = -\frac{\pi}{2} - 2(-1) = 2 - \tfrac{\pi}{2} > 0$$

$$f(\tfrac{\pi}{2}) = \tfrac{\pi}{2} - 2\sin(\tfrac{\pi}{2}) = \tfrac{\pi}{2} - 2(1) = \tfrac{\pi}{2} - 2 < 0$$

Since we have opposite sign from IVT there is $c \in (-\tfrac{\pi}{2}, \tfrac{\pi}{2})$ such that

$$f(c) = c - 2\sin c = 0$$

$$\Rightarrow \frac{c}{2} = \sin c$$

Q6. Find the equation of the tangent line to the graph of $f(x) = \frac{1}{x+1}$ at $x = 1$.

$$m = f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{\frac{1}{1+h+1} - \frac{1}{1+1}}{h}$$

$$= \lim_{h\to 0} \frac{2 - (2+h)}{(h)(2)(2+h)} = \lim_{h\to 0} \frac{-\cancel{h}}{\cancel{h}(2)(2+h)}$$

$$= -\frac{1}{4}$$

$f(1) = 1/2$

$(x_0, y_0) = (1, 1/2) \qquad m = -1/4$

The equation is

$$y - \frac{1}{2} = -\frac{1}{4}(x - 1)$$

$$y = -\frac{1}{4}x + \frac{3}{4}$$

Q3. If $\frac{\sqrt{4+x}-2}{x} \le f(x) \le \frac{x\csc 4x}{\cos x}$ find the value of $\lim_{x\to 0} f(x)$

(1) $$\lim_{x\to 0}\frac{\sqrt{4+x}-2}{x} = \lim_{x\to 0}\frac{\sqrt{4+x}-2}{x}\cdot\frac{\sqrt{4+x}+2}{\sqrt{4+x}+2}$$

$$= \lim_{x\to 0}\frac{4+x-4}{x(\sqrt{4+x}+2)} = \frac{1}{4}$$

(2) $$\lim_{x\to 0}\frac{x\csc 4x}{\cos x} = \lim_{x\to 0}\frac{4x}{4\sin 4x}\cdot\frac{1}{\cos x} = \frac{1}{4}(1)(1)$$

from the squeezing th$^{\circ}$.

$$\lim_{x\to 0} f(x) = \frac{1}{4}.$$

Q4. If $f(x) = x|x-2|$ then find $f'(2)$

$$f(x) = \begin{cases} x^2 - 2x & x \ge 2 \\ 2x - x^2 & x < 2 \end{cases}$$

$$f'_+(2) = \lim_{h\to 0^+}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0^+}\frac{f(2+h)-f(2)}{h} = \lim_{h\to 0}\frac{(2+h)^2-2(2+h)-0}{h}$$

$$= \lim_{h\to 0}\frac{\cancel{4}+\cancel{4h}+h^2-\cancel{4}-2h}{h}$$

$$= \lim_{h\to 0}\frac{h(2+h)}{h} = 2$$

$$f'_-(2) = \lim_{h\to 0^-}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0^-}\frac{2(2+h)-(2+h)^2-0}{h} = \lim_{h\to 0}\frac{\cancel{4}+2h-\cancel{4}-4h-h^2}{h}$$

$$= \lim_{h\to 0}\frac{-2h+h^2}{h} = \lim_{h\to 0}\frac{h(-2+h)}{h}$$

$$= -2$$

$f'(2)$ DNE