Each question worth 10 points

King Fuhd University of Petroleum and Minerals
Department of Mathematical Sciences

Math 101-1
First Exam , Semester 013
Time:3:00-4:40, Tuesday, July 2 ;2002

100

Name :--ID # :--

Q1. Use the definition of derivative to find $f'(x)$ where $f(x) = \sqrt[2]{x+1}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h+1} - \sqrt{x+1}}{h} \cdot \frac{\sqrt{x+h+1} + \sqrt{x+1}}{\sqrt{x+h+1} + \sqrt{x+1}}$$

$$= \lim_{h \to 0} \frac{\cancel{x} + h + \cancel{1} - \cancel{x} - \cancel{1}}{h\left(\sqrt{x+h+1} + \sqrt{x+1}\right)}$$

$$= \frac{1}{2\sqrt{x+1}}$$

Q2. State and then prove the Quotient Derivative Rule (Division Rule).

Seethe book

Q3. Find the limit if it is exist $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$

$$\lim \frac{1-\cos x}{x^2} \cdot \frac{1+\cos x}{1+\cos x} = \lim_{x \to 0} \frac{1-\cos^2 x}{x^2(1+\cos x)}$$

$$= \lim_{x \to 0} \frac{\sin^2 x}{x^2} \cdot \lim_{x \to 0} \frac{1}{1+\cos x}$$

$$= 1^2 \cdot \frac{1}{2} = \frac{1}{2}.$$

Q4. Find the limit if it is exist $\lim_{x \to \infty} \sqrt{x^2 + x + 1} - x$ $\quad , \frac{\sqrt{x^2+x+1} + x}{\sqrt{x^2+x+1} + x}$

$$= \lim_{x \to \infty} \frac{\cancel{x^2} + x + 1 - \cancel{x^2} \quad / x}{\sqrt{x^2+x+1} + x \quad / x}$$

$$= \lim_{x \to \infty} \frac{1 + \frac{1}{x}}{\sqrt{1 + \frac{1}{x} + \frac{1}{x^2}} + 1} = \frac{1+0}{\sqrt{1+0+0} + 1}$$

$$= 1/2.$$

Q5. Find Derivative of $f(x) = \frac{x+1}{\sqrt{x}} + \sqrt[5]{\pi^2}$

$$f'(x) = \frac{(1)\sqrt{x} - (x+1)\frac{1}{2\sqrt{x}}}{(\sqrt{x})^2} + 0 = \frac{2x - x - 1}{2x\sqrt{x}}$$

$$= \frac{x-1}{2x\sqrt{x}}$$

Q6. Find $f'(x), f''(x)$ and $f'''(x)$ if $f(x) = \dfrac{x-1}{x^2}$

$$f'(x) = \frac{x^2 - 2x(x-1)}{x^4} = \frac{x^2 - 2x^2 + 2x}{x^4} = \frac{x[2-x]}{x^4} = \frac{2-x}{x^3}$$

$$f''(x) = \frac{-x^3 - (2-x)3x^2}{x^6} = \frac{x^2[-x-6+3x]}{x^6} = \frac{2x-6}{x^4}$$

$$f'''(x) = \frac{2x^4 - 4x^3(2x-6)}{x^8} = \frac{x^3(2x-8x+24)}{x^8} = \frac{24-6x}{x^5}$$

Q7. Find k if the curve $y = x^2 + k$ is tangent to the line $y = 2x$

let (x_0, y_0) be the point where $y = x^2+k$ is tangent to $y = 2x$

Now the slope of the curve is $y' = 2x \Big|_{x_0} = 2x_0 \longrightarrow y = x^2 + k$

and the slope of the line is $2 \longrightarrow y = 2x$

at (x_0, y_0) therefore $2x_0 = 2$ So $x_0 = 1$

But (x_0, y_0) on the line So $y_0 = 2x_0 = 2$

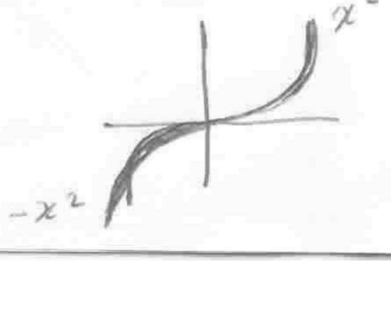
Also (x_0, y_0) is on the curve $y_0 = x_0^2 + k$

$\Rightarrow k = y_0 - x_0^2 = 2 - (1)^2 = 1$

Q8. If $f(x) = x|x|$ then find $f'(0)$

$$f(x) = \begin{cases} x^2 & x \geq 0 \\ -x^2 & x < 0 \end{cases}$$

$$f'(x) \begin{cases} 2x \\ -2x \end{cases} \qquad f'_-(0) = f'_+(0) = 0$$



$$f'_-(0) = \lim_{h \to 0^-} \frac{-h^2}{h} = 0$$

$$f'_+(0) = \lim_{h \to 0^+} \frac{h^2}{h} = 0$$

Q9. Find Δy for $f(x) = x^3 + x^2 - 1$ if x changes from 2 to 2.1 then find the rate of change.

$$\Delta y = f(2.1) - f(2) = (2.1)^3 + (2.1)^2 - 1 - 2^3 - 2^2 + 1$$
$$= 9.261 + 4.41 - 8 - 4$$
$$= 1.671$$

$$\text{rate of change} = \frac{\Delta y}{\Delta x} = \frac{1.671}{0.1} = 16.71$$

Q10. If $f(x)$ is continuous function with $f(0) = 2$ and $g(x) = \dfrac{x^2 - \tan^2 3x}{\sin^2 x}$ then find $\lim\limits_{x\to 0}[f(x) + g(x)]$

$f(x)$ is continuous $\Rightarrow$ $f(0) = \lim\limits_{x\to 0} f(x) = 2$.

Now $\lim\limits_{x\to 0}[f(x) + g(x)] = \lim\limits_{x\to 0} f(x) + \lim\limits_{x\to 0} g(x) = \boxed{2 + (-8) = -6}$

$$\lim_{x\to 0} g(x) = \lim_{x\to 0} \frac{x^2}{\sin^2 x} - \lim_{x\to 0}\frac{\tan^2 3x}{\sin^2 x} = 1^2 - 3^2 \lim_{x\to 0} \frac{\sin^2 3x}{(3x)^2} \cdot \frac{x^2}{\sin^2 x} \cdot \frac{1}{\cos^2 3x}$$

$$= 1 - 9 \lim_{x\to 0}\left(\frac{\sin 3x}{3x}\right)^2 \cdot \lim_{x\to 0}\left(\frac{x}{\sin x}\right)^2 \cdot \lim_{x\to 0}\left(\frac{1}{\cos 3x}\right)^2$$

$$= 1 - 9 \cdot 1 \cdot 1 = 1 - 9 = -8$$

$\lim\limits_{x\to 0} h(x) \to -6$