

2.5 Continuity

graphically connecting plot.

Definition A function f is continuous at number

a if

③ $\lim_{x \rightarrow a} f(x) = f(a)$

This means ① $f(a)$ defined

② $\lim_{x \rightarrow a} f(x)$ exists

if not then it is discontinuous.

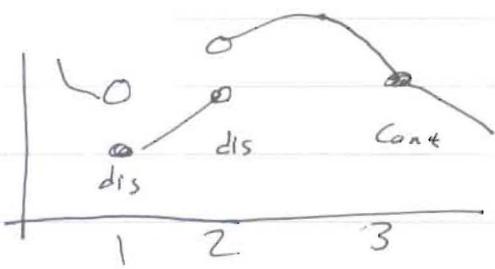
End points

$f(x)$ continuous at a left of a if

$f(a) = \lim_{x \rightarrow a^+} f(x)$, $f(x)$, , right of b if

$\lim_{x \rightarrow b^-} f(x) = f(b)$

Example



Ex. $f(x) = \begin{cases} \frac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 3 & \text{if } x = 2 \end{cases}$

$\lim_{x \rightarrow 2} \frac{(x-2)(x+1)}{x-2} = 3 = f(2)$

Def. A function f is continuous from the right at a if

$$\lim_{x \rightarrow a^+} f(x) = f(a)$$

f cont. from left if

$$\lim_{x \rightarrow a^-} f(x) = f(a)$$

$$f(x) = \lfloor x \rfloor$$

$$\lim_{x \rightarrow n^+} f(x) = f(n) = n$$

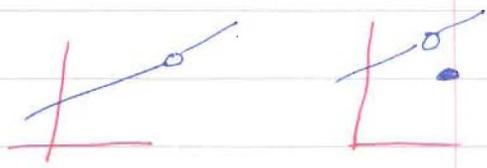
while

$$\lim_{x \rightarrow n^-} f(x) \neq f(n) = n - 1.$$

Ex $f(x) = [x]$ is discontinuous at all integer numbers. (2/1)

Four
Three type of discontinuities

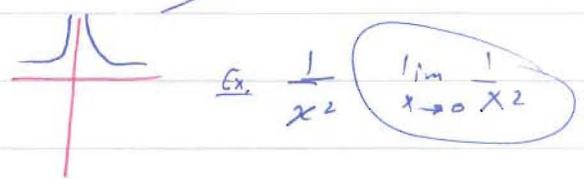
① Removable dis



② Jump dis



③ Infinite dis



④ Oscillating dis



Continuous function

f(x) continuous in interval if it continuous at each point in the interval.

Ex $f(x) = \frac{1}{x}$ is continuous function on its domain.
not continuous at 0 but it is not in the domain.

Ex. $\lim_{x \rightarrow x} x = x = f(x) \quad \forall c \in \mathbb{R}$ $f(x) = x \quad f(x) = k$
 $\lim_{x \rightarrow c} k = k = f(c) \quad \forall c \in \mathbb{R}$ \Rightarrow both continuous
 \forall all real numbers

Ex. $f(x) = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$ cont $\forall x \in \mathbb{R}$.

Th^o

Continuous function properties.

If $f(x)$ & $g(x)$ continuous at $x=c$. then
 $f+g$, $f-g$, kf , $\forall k$, $f \cdot g$, f/g $g(c) \neq 0$
 f^n , n positive integer $\leftrightarrow \sqrt[n]{f}$

These rules are true because

of the limit. given $\lim_{x \rightarrow c} f(x) = f(c)$; $\lim_{x \rightarrow c} g(x) = g(c)$

$$\lim_{x \rightarrow c} f(x) + \lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} f(x) + \lim_{x \rightarrow c} g(x)$$

$$= f(c) + g(c) = (f+g)(c)$$

Every polynomial and rational function are continuous function except when $g(c) \neq 0$

Ex. $f(x) = \frac{x-2}{x+1}$ cont. except at $\underline{\underline{x=-1}}$

Graphically it continuous line

Curves discont. when they have breaks or holes.

Ex $\sin \theta$, $\cos \theta$ are continuous everywhere.

$$\tan x = \frac{\sin x}{\cos x} \text{ cont except when } \cos x = 0.$$

Inverse functions

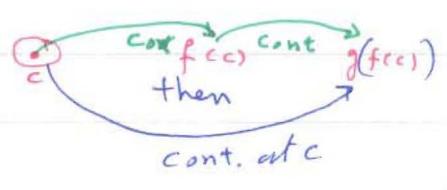
If $f(x)$ is continuous then the inverse is also continuous

Clear from the fact that f, f^{-1} are symmetric across the line $y=x$.

So since all trigonometric are continuous over their domains then also their inverse.

Composites

All composites of continuous functions is continuous.



Th If f is cont. at c & g is cont. at $f(c)$ then $g \circ f$ is cont. at c .

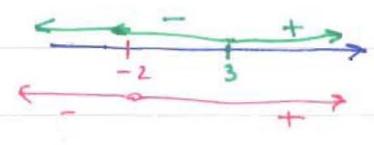
Show that these functions are cont. in their domain.

Ex. (a) $y = \sqrt{x^2 - 2x - 5}$

$g(x) = \sqrt{x}$ is cont. $[0, \infty)$

$f(x) = x^2 - 2x - 5$ when $x^2 - 2x - 5 > 0$ $f \circ g$ Compositon

$(x - 3)(x + 2) > 0$



$f(x) \geq 0$ if $x \in (-\infty, -2] \cup [3, \infty)$

$$(b) y = \frac{x^{2/3}}{1+x^2} = \frac{\sqrt[3]{x^2}}{1+x^2} \quad f(x) = x^2 \quad g(x) = \sqrt[3]{x} \quad (27)$$

composites

quotient \rightarrow denominator $\rightarrow 0$

continuous everywhere

y cont $\forall x \in \mathbb{R}$

$$(c) y = \left| \frac{x-1}{x^2-2} \right|$$

$$f(x) = |x|$$

cont. function

$$x \neq \pm\sqrt{2}$$

y cont $\forall x \in \mathbb{R}$ except $\pm\sqrt{2}$.

$$(d) y = \left| \frac{x \sin x}{x^2+1} \right| \quad \text{continuous } \forall x \in \mathbb{R}.$$

The limit of cont. function

If g is cont. at b and $\lim_{x \rightarrow c} f(x) = b$
 then

$$\lim_{x \rightarrow c} g(f(x)) = g(b) = g(\lim_{x \rightarrow c} f(x)) = g(b).$$

Basically you can take the limit inside
 if g cont. & f limit exist.

limit symbol can be moved
 through a continuous function sign
 if the limit exist

~~(2)~~ 2 conditions cont }
 exist }

Ex. (a) $\lim_{x \rightarrow \sqrt{2}} \cos(x^2 - 2) = \cos \lim_{x \rightarrow \sqrt{2}} x^2 - 2 = \cos 0 = 1$

(b) $\lim_{x \rightarrow 2} \sin^{-1} \left(\frac{2-x}{4-x^2} \cdot 2 \right) = \sin^{-1} \left(\lim_{x \rightarrow 2} \frac{2-x}{(2-x)(2+x)} \cdot 2 \right)$

$$= \sin^{-1} \frac{1}{4} \cdot 2 = \sin^{-1} \frac{1}{2} = \frac{\pi}{6}$$

(c) $\lim_{x \rightarrow 0} \sqrt{x+1} e^{\tan x}$

$$= \lim_{x \rightarrow 0} \sqrt{x+1} \cdot \lim_{x \rightarrow 0} e^{\tan x}$$

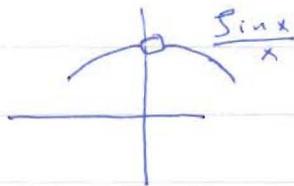
$$= \sqrt{0+1} \cdot e^0 = 1.$$

Continuity extension to a point (Removable dis.)

Ex. $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

but at $x=0$

the function is not defined



So we can extend the domain of $f(x)$ to make $f(x)$ continuous at $x=0$

$$f(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

Now $f(x)$ is continuous at $x=0$ and in all the real numbers.

Ex. Show that $f(x) = \frac{x^2 + x - 6}{x^2 - 4}$ $x \neq 2$

has a continuous extension at $x=2$
and find that extension

Sol. $f(x) = \frac{(x-2)(x+3)}{(x-2)(x+2)} = \frac{x+3}{x+2}$

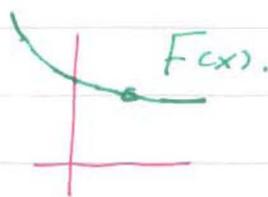
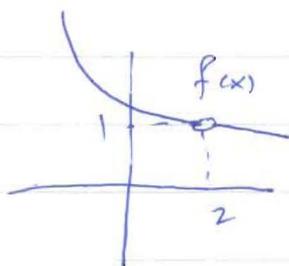
So the $\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} \frac{x+3}{x+2} = \frac{5}{4}$

the function is not defined at $x=2$.

So we can make $f(x)$ continuous at $x=2$

$F(x)$ is the continuous extension of f at $x=2$

$$F(x) = \begin{cases} \frac{x^2 + x - 6}{x^2 - 4} = \frac{x+3}{x+2} & x \neq 2 \\ \frac{5}{4} & x = 2 \end{cases}$$



Ex. Find the continuity of interval of $f(x) = \frac{\ln x + \tan^{-1} x}{x^2 - 1}$

Denominator $\ln x$ $\boxed{x > 0}$, $\tan^{-1} x$ \mathbb{R}
Denominator $x^2 - 1 \neq 0$ $x \neq 1$
 $x \neq -1$

the intersection $[0, 1) \cup (1, \infty)$

The following functions are continuous in their Domain

Polynomials, rational functions
 root function, trigonometric function
 inverse trigonometric, exponential function
 Logarithmic functions.

$$\text{Ex. } \lim_{x \rightarrow \pi} \left(\frac{\sin x}{2 + \cos x} \right) = \frac{\sin \pi}{2 + \cos \pi} = \frac{0}{2 + (-1)} = 0$$

$$f(\pi) = 0$$

$$\text{Ex. } \lim_{x \rightarrow 1} \sin^{-1} \left(\frac{1 - \sqrt{x}}{1 + x} \right) \quad \sin^{-1} \text{ cont.}$$

$$\sin^{-1} \lim_{x \rightarrow 1} \frac{(1 - \sqrt{x})(1 + \sqrt{x})}{1 + x} = \sin^{-1} \frac{1}{2} = \pi/6.$$

Ex. find discontinuity points

$$f(x) = \begin{cases} \frac{x^2 - 9}{x - 3} & x \neq 3 \\ 4 & x = 3 \end{cases}$$

$$\lim_{x \rightarrow 3} f(x) = 6 \neq 4$$

So $f(x)$ dis cont.
only at $x = 3$.

should go after that definition (each) Prove that $f(x)$ cont. in its domain (31)

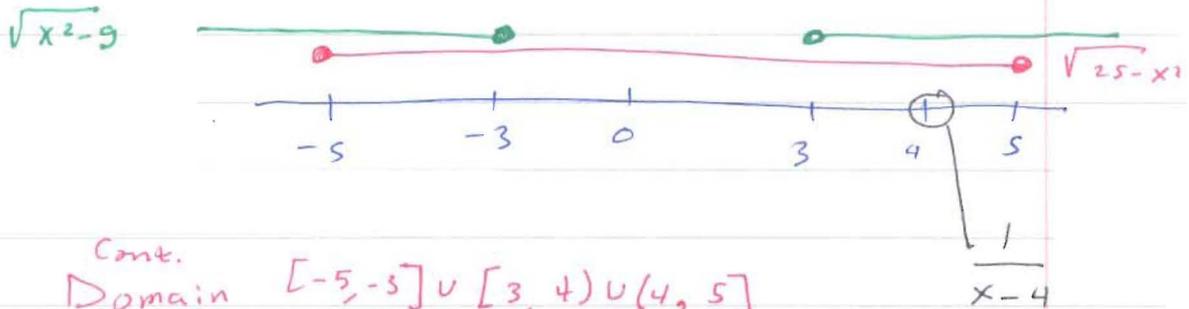
Ex. $f(x) = \sqrt{1-x^2}$ $[-1, 1]$

for $(-1, 1)$ $\lim_{x \rightarrow c} f(x) = \sqrt{\lim_{x \rightarrow c} 1-x^2}$ $c \in (-1, 1)$
 $= \sqrt{1-c^2} = f(c)$

Left $+1$ $\lim_{x \rightarrow 1^-} \sqrt{1-x^2} = 0 = f(1)$

Right -1 $\lim_{x \rightarrow -1^+} \sqrt{1-x^2} = 0 = f(-1)$

Ex. $f(x) = \frac{\sqrt{x^2-9} \sqrt{25-x^2}}{x-4}$



Cont. Domain $[-5, -3] \cup [3, 4) \cup (4, 5]$

Ex. Find the value of k so $f(x)$ is cont. at $x=0$

$$f(x) = \begin{cases} \frac{\sin x}{x} & x < 0 \\ 3x + 2k^2 & x \geq 0 \end{cases}$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} 3x + 2k^2 = 2k^2 = f(0)$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{\sin x}{x} = 1$$

So $2k^2 = 1 \Rightarrow k = \pm \frac{1}{\sqrt{2}}$

(32)

Ex. If $f(x) = \begin{cases} 2x-3 & x < 2 \\ c & x = 2 \\ x^2 & x > 2 \end{cases}$

Is there a value of c that makes f cont.

(is this removable dis)

Ex.

no

$\lim_{x \rightarrow 2^-} f(x) = 1$ while $\lim_{x \rightarrow 2^+} f(x) = 4$

$f(x) = \begin{cases} 3x-9 & x < 1 \\ b & x = 1 \\ x^2+1 & x > 1 \end{cases}$

find a & b such that f is cont.

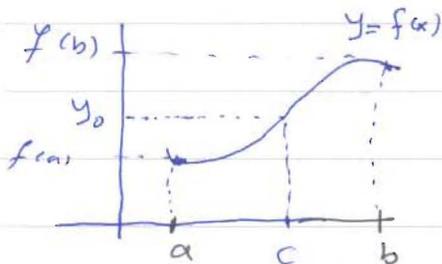
$\lim_{x \rightarrow 1^+} x^2+1 = 2$ $f(1) = b$

$\lim_{x \rightarrow 1^-} 3x-9 = 3-9$

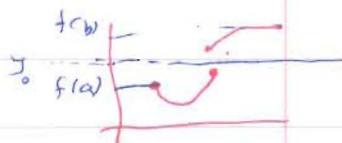
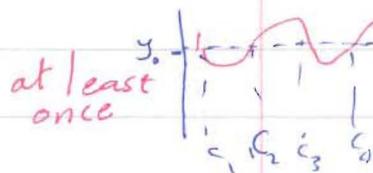
$b = 2, \quad a = 1.$

Intermediate Value th of continuous functions

Th If f is a continuous function on a closed interval $[a, b]$ and if y_0 is any value between $f(a)$ and $f(b)$, then there exist a number $c \in (a, b)$ such that $y_0 = f(c)$.



Continuity is essential



A consequence of for root finding

Some books make it a th^o.

If $f(x)$ cont. on $[a, b]$ & if $f(a)$ & $f(b)$ are nonzero and have opposite sign, then there exist at least one solution to the equation $f(x) = 0$.

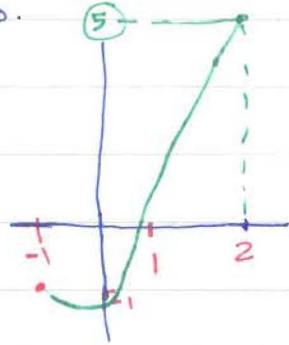
Ex. Show the roots of $f(x) = x^4 + 2 - 1$ $[-1, 2]$

① $f(x)$ cont. (poly.)

② $f(-1) = -1$ $f(2) = 5$ opposite sign

then from IMVT $\exists c \in (-1, 2)$

such that $f(c) = 0$.



Also if $[-1, 1]$

$$f(-1) = -1 \quad f(1) = 1$$

then if $[0, 1]$

$$f(0) = -1 \quad f(1) = 1$$

if $[0.5, 1]$

$$f(0.5) = 0.0625 + 0.5 - 1 = -0.4375 \quad f(1) = 1$$

if $[0.5, 0.75]$

$$f(0.5) = -0.4375 \quad f(0.75) = 0.3164 + 0.75 - 1 = +0.0664$$

Ex. Using IMVT. show that eq. $x^3 - 4x + 1$ has a solution between 1 and 2.

$f(x) = x^3 - 4x + 1$ is a polynomial

$$f(1) = -2 \text{ while } f(2) = 1$$

$$f(1) < 0 < f(2)$$

From IMVT $\exists c \in (1, 2)$ such that $f(c) = 0$

Ex. $\sqrt{2x+5} = 4 - x^2$ Using IMVT prove that the equation has a solution

$$\text{let } f(x) = \sqrt{2x+5} + x^2 - 4 = 0$$

Now $\sqrt{2x+5}$ is cont. in the Domain $[-\frac{5}{2}, \infty)$

$x^2 - 4$ cont. poly.

by trial & error So $f(x)$ cont. $[-\frac{5}{2}, \infty)$

$$f(0) = -1.76$$

$$f(2) = 3$$

$f(x)$ is cont. $[0, 2] \subset [-\frac{5}{2}, \infty)$ cont by IMVT

there is $c \in (0, 2)$ such that $f(c) = 0$