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## 2.3 Calculating limits Using the limit laws

### limit laws

Suppose  $\lim_{x \rightarrow a} f(x) \text{ and } \lim_{x \rightarrow b} g(x) \text{ exist}$

then

c: constant

1. Sum Rule

$$\lim_{x \rightarrow a} (f(x) + g(x)) = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$$

2. Difference Rule  $\lim_{x \rightarrow a} (f(x) - g(x)) = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$

3. Constant multiple Rule  $\lim_{x \rightarrow a} (c f(x)) = c \lim_{x \rightarrow a} f(x)$

4. Product Rule  $\lim_{x \rightarrow a} (f(x) \cdot g(x)) = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$

5 Quotient Rule

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$$

If  $\lim_{x \rightarrow a} g(x) \neq 0$

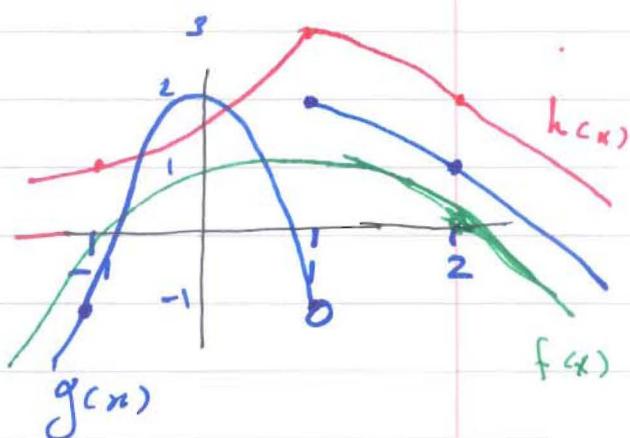
Ex. ④  $\lim_{x \rightarrow 2} (f(x) + 3g(x)) / h(x) = \frac{(0 + 3(1))}{2} = \frac{3}{2}$

⑤  $\lim_{x \rightarrow 1} \frac{f(x) \cdot g(x)}{h(x)} \leftarrow \text{DNE}$

⑥  $\lim_{x \rightarrow -1} [f(x) - g(x)] g(x)$

$$= (0 - (-1)) 1$$

$$= 1$$

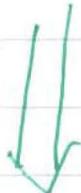


## Rules

$$\lim_{x \rightarrow a} [f(x)]^n = \left[ \lim_{x \rightarrow a} f(x) \right]^n \quad n \text{ positive integer}$$

if  $\lim_{x \rightarrow a} f(x)$  exists

$$\lim_{x \rightarrow a} c = c$$



$$\lim_{x \rightarrow a} x^n = a^n$$

$$\lim_{x \rightarrow a} \sqrt[n]{x} = \sqrt[n]{a}$$

n positive integer  
if n even,  $a > 0$ .



$$\lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)}$$

$\lim_{x \rightarrow a} f(x) > 0$

Ex.  $\lim_{x \rightarrow -2} 3x^2 - x - 1 = 3(-2)^2 - (-2) - 1$

Ex.  $\lim_{x \rightarrow 1} \frac{x^3 + 2x + 1}{5 - 2x}$

## Direct Substitution Property

14

The limit of polynomials

If  $P(x) = a_n x^n + \dots + a_0$ , then

$$\lim_{x \rightarrow c} P(x) = P(c) = \dots$$

The limit of rational function

If  $P(x)$  &  $Q(x)$  are polynomials &

$Q(c) \neq 0$ , then

$$\lim_{x \rightarrow c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}$$

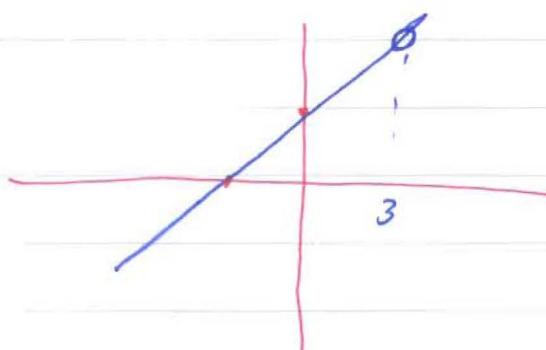
Ex  $\lim_{x \rightarrow -1} \frac{x^3 - 4x + 1}{x^2 + 1} =$

Now techniques common factors

Eliminating zero denominators Algebraically,

above theory apply when there is no zero

Ex  $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3} = \cancel{\lim_{x \rightarrow 3} x^2 - 8} - \lim_{x \rightarrow 3} \frac{x - \cancel{3}(x+3)}{x - \cancel{3}} = 6$



If  $f(x) = g(x)$   
when  $x \neq a$

then  $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x)$

Ex  $\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 100} - 10}{x^2}$

analytically

or

$$\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 100} - 10}{x^2} \cdot \frac{\sqrt{x^2 + 100} + 10}{\sqrt{x^2 + 100} + 10}$$

use calculator

the same way

$x \rightarrow 0$  from left

& right

$$= \lim_{x \rightarrow 0} \frac{1}{\sqrt{x^2 + 100} + 10} = \frac{1}{20} = 0.05.$$

## The Sandwich Th<sup>o</sup>

If  $g(x) \leq f(x) \leq h(x)$  for  $x$  near  $c$  {except possibly at  $x=c$ }

$$\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} h(x) = L$$

Then  $\lim_{x \rightarrow c} f(x) = L$

Ex.  $\lim_{x \rightarrow 0} x^2 \sin \frac{1}{x}$  not true if  $\lim_{x \rightarrow 0} x^2 \frac{\sin \frac{1}{x}}{\text{DNE}}$

$$-\leq \sin \frac{1}{x} \leq 1$$

$$x^2 \geq 0 \quad \lim_{x \rightarrow 0} -x^2 = \lim_{x \rightarrow 0} x^2 = 0$$

by the sandwich th<sup>o</sup>

$$\lim_{x \rightarrow 0} x^2 \sin \frac{1}{x} = 0$$

not the definition  
 $\lim_{x \rightarrow 0} x^2 \sin \frac{1}{x}$

16

Ex Same with ①  $\lim_{\theta \rightarrow 0} \sin \theta = 0$

$$-|\theta| \leq \sin \theta \leq |\theta|$$

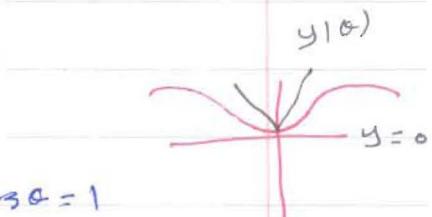
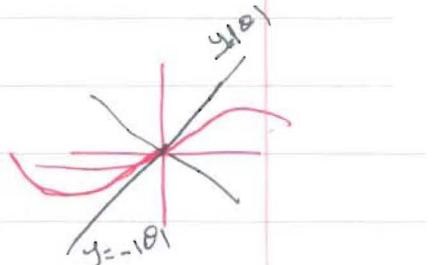
$$\textcircled{2} \lim_{\theta \rightarrow 0} \cos \theta = 1$$

$$0 \leq 1 - \cos \theta \leq |\theta|$$

$$\lim_{\theta \rightarrow 0} 1 - \cos \theta = 0 \Rightarrow \lim_{\theta \rightarrow 0} \cos \theta = 1$$

$$\textcircled{3} \lim_{x \rightarrow c} |f(x)| = 0 \Rightarrow \lim_{x \rightarrow c} f(x) = 0$$

$$\text{Since } -|f(x)| \leq f(x) \leq |f(x)|$$



Th If  $f(x) \leq g(x)$  for  $x$  near  $c$  except  $x=c$   
 $\lim_{x \rightarrow c} f(x) \leq \lim_{x \rightarrow c} g(x)$  (both limits exist)

Ex. Find the  $\lim_{x \rightarrow 0} \tan x$

$$= \lim_{x \rightarrow 0} \frac{\sin x}{\cos x}$$

Ex

$$\text{Find } \lim_{x \rightarrow 1} f(x) \quad f(x) = \begin{cases} x & \text{if } x \neq 1 \\ 3 & \text{if } x = 1 \end{cases}$$

$$f(1) = 3 \quad \lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} x = 1$$

17

$$\text{Ex. } \lim_{h \rightarrow 0} \frac{(1+h)^2 - 1}{h} \left( \frac{0}{0} \right) = \lim_{h \rightarrow 0} \frac{1+2h+h^2 - 1}{h}$$

$$= \lim_{h \rightarrow 0} (2+h) = 2.$$

$$\text{Ex. } \lim_{t \rightarrow 0} \frac{\sqrt{t^2+9} - 3}{t^2} \cdot \frac{\sqrt{t^2+9} + 3}{\sqrt{t^2+9} + 3} \cdots = \frac{1}{6}.$$

$$\text{Ex. } \lim_{x \rightarrow 0} \left( \frac{\sqrt{x^2+4} - 2}{x} \right)^2 \times \sqrt{x^2+4} + 2$$

$$\text{Ex. } \lim_{x \rightarrow 0} \left( \frac{1}{x^2} \right) \left( \frac{1}{x^2} \right) = \lim_{x \rightarrow 0} \frac{x^2 - 1}{(x^2)^2} =$$

$$\begin{aligned} \text{Ex. } \lim_{t \rightarrow 1} \frac{t^3 + t^2 - 5t + 3}{t^3 - 3t + 2} \left( \frac{0}{0} \right) &= \lim_{t \rightarrow 1} \frac{(t-1)(t^2+2t-3)}{(t-1)(t^2+t-2)} \\ &= \lim_{t \rightarrow 1} \frac{(t-1)^2(t+3)}{(t-1)^2(t+2)} = 4/3 \end{aligned}$$

$$\text{Ex. } \lim_{x \rightarrow 0} |x| = 0$$

$$\text{Ex. } \lim_{x \rightarrow 0} \frac{|x|}{x} \quad \text{DNE}$$

$$\text{Ex. } f(x) = \sqrt{9-x^2} \quad \text{lim in the domain}$$

Ex. The greatest integer function

$$f(x) = \llbracket x \rrbracket \quad \llbracket -\frac{1}{2} \rrbracket = -1$$

$$\lim_{x \rightarrow 3^+} \llbracket x \rrbracket = 3 \quad \lim_{x \rightarrow 5^-} \llbracket x \rrbracket = 2$$