

2.8 The Derivative as a Function

Def The derivative of the function $f(x)$ w.r.t. x is the function f'

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

provided the limit exists.

Alt. Def. $f'(x) = \lim_{z \rightarrow x} \frac{f(z) - f(x)}{z - x}$

Calculating derivative from definition

Ex. $f(x) = \frac{x+2}{x}$ $f(x+h) = \frac{x+h+2}{x+h}$

$$f'(x) = \lim_{h \rightarrow 0} \frac{\frac{x+h+2}{x+h} - \frac{x+2}{x}}{h} = \lim_{h \rightarrow 0} \frac{x(x+h+2) - (x+2)(x+h)}{h(x)(x+h)}$$

$$= \lim_{h \rightarrow 0} \frac{x^2 + xh + 2x - x^2 - xh - 2x - 2h}{h(x)(x+h)}$$

$$= \lim_{h \rightarrow 0} \frac{-2h}{h(x)(x+h)} = \frac{-2}{x^2}$$

Def A function f is differentiable at a if $f'(a)$ exists. It is diff on open interval (a, b) , (a, ∞) or $(-\infty, a)$ if it is differentiable at every point.

Ex $f(x) = x^3 + x$

$$f'(x) = \lim_{h \rightarrow 0} \frac{(x+h)^3 + (x+h) - x^3 - x}{h} = 3x^2 + 1$$

Eq.

Find the tangent to the curve at $x=1$

Point } $y = mx + b$ or $(y - y_0) = m(x - x_0)$
 slope }

Point $x=1 \Rightarrow y=2$ the slope at $x=1$ $f'(1) = 4$

then the tangent Eq

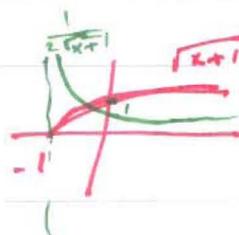
$$y - 2 = 4(x - 1)$$

$$y = 4x - 2$$

Ex. $f(x) = \sqrt{x+1}$ find the Eq. of tangent line at $(8, 3)$

$$f'(x) = \lim_{h \rightarrow 0} \frac{\sqrt{x+h+1} - \sqrt{x+1}}{h} \cdot \frac{\sqrt{x+1} + \sqrt{x+1}}{\sqrt{x+1} + \sqrt{x+1}} = \frac{1}{2\sqrt{x+1}}$$

$$f'(8) = \frac{1}{6}$$



Eq. $y - 3 = \frac{1}{6}(x - 8)$

Ex. $f(x) = x^2 - 8x + 9$ find $f'(a)$

$$f'(a) = \lim_{h \rightarrow 0} \frac{(a+h)^2 - 8(a+h) + 9 - a^2 + 8a - 9}{h}$$

$$= 2a - 8.$$

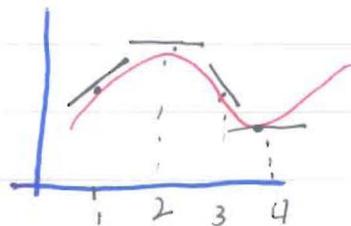
Notations for derivative

$$f'(x), \frac{dy}{dx} = y' = \frac{df}{dx}, D(f)(x), D_x(f)$$

$$f'(x_0) = \left. \frac{df}{dx} \right|_{x=x_0}$$

Graphing the derivative

Ex.



$$\begin{aligned} f'(1) &= 1 \\ f'(2) &= 0 \\ f'(3) &= -1 \\ f'(4) &= 0 \end{aligned}$$

Ex Match the graph a-f with their derivatives A-F

	a	b	c	d	e	f
f						
f'	A	B	C	D	E	F

One-sided derivatives

$$f'_+(x) = \lim_{h \rightarrow 0^+} \frac{f(x+h) - f(x)}{h}$$

(51)

$$f'_-(x) = \lim_{h \rightarrow 0^-} \frac{f(x+h) - f(x)}{h}$$

Ex.

$$f(x) = \begin{cases} x^2 + 2 & x \leq 1 \\ x + 2 & x > 1 \end{cases} \quad \text{cont but not diff.}$$

$$f(1) = 3 = \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} x + 2 = 3 = \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} x^2 + 2 = 3$$

So $f(x)$ is cont.

Now

$$f'_+(1) = \lim_{h \rightarrow 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \rightarrow 0^+} \frac{1+h+2 - 1+2}{h} = 1$$

while

$$\begin{aligned} f'_-(1) &= \lim_{h \rightarrow 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \rightarrow 0^-} \frac{(1+h)^2 + 2 - (1^2 + 2)}{h} \\ &= \lim_{h \rightarrow 0^-} \frac{h(2+h)}{h} = 2 \end{aligned}$$

Not differentiable.

Ex $f(x) = |x|$ Cont. but not diff. at $x=0$.

$$f'_+(0) = 1 \quad \text{while} \quad f'_-(0) = -1$$

Differentiability

points of nondifferentiability when the curve does not have a tangent line

Type

- Corners

Ex. $f(x) = |x|$



- Vertical tangent / cusp

Ex. $f(x) = \sqrt[3]{x}$



Ex. $f(x) = \sqrt{x^2}$



- points of discontinuity

Ex.

$$f(x) = \lfloor x \rfloor$$

Th^o If $f(x)$ is diff then it is Conc.

Proof.

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Now

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} (f(a) + f(x) - f(a))$$

$$= \lim_{x \rightarrow a} f(a) + \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \cdot (x - a)$$

$$= f(a) + f'(a) \cdot (0) = f(a).$$

Ex. Find the equation of the tangent line to the graph of $y=f(x)$ at the point $x=3$;
 Given $f(3)=-1, f'(3)=5$

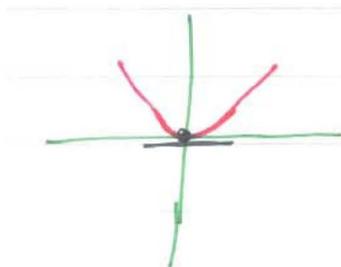
$$f'(3) = 5 = m \text{ (the slope)} \quad y+1 = 5(x-3)$$

Ex. Find $f(x)$ and a if $f'(a) = \lim_{h \rightarrow 0} \frac{\cos(\pi+h) + 1}{h}$

Now $\cos \pi = -1$ so $f'(a) = \lim_{h \rightarrow 0} \frac{\cos(\pi+h) - \cos(\pi)}{h}$

$$\Rightarrow a = \pi \quad \& \quad f(x) = \cos x$$

Ex. Sketch the graph of a function f for which
 $f(0)=0, f'(0)=0$ $f'(x) > 0$ if $x > 0$
 $f'(x) < 0$ if $x < 0$



Higher Derivatives

y'	y''	y'''	$y^{(4)}$
$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$		$\frac{d^4y}{dx^4}$

$$\left| \begin{aligned} \frac{d^2y}{dx^2} &= \frac{d}{dx} \cdot \frac{dy}{dx} \\ &= s(t) \\ \cdot \quad s'(t) &= v(t) \\ s''(t) &= v'(t) = a(t) \end{aligned} \right.$$

Ex. Find f'' if $f(x) = x^3 - x$, Use limit then find $f^{(4)}$