The Principle of Mathematical Induction

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Outline

Dominoes

- Suppose you have some (finite) number of dominoes lined up and satisfying the following condition:
  - as each domino falls it knocks over the domino next in line; or
  - the fall of domino n implies the fall of domino $n + 1$.
- Q: If we cause the first domino to fall over will all the dominoes fall over?
- A: It seems fairly obvious that the answer is yes, they will all fall over.

- Now, imagine that instead of a finite line of dominoes we have an infinite line still satisfying the same condition
  - ie. the fall of domino n implies the fall of domino $n + 1$.
- Suppose we again cause the first domino to fall.
- Q: Do all the dominoes fall over?
- Less obvious?
- The principle of mathematical induction asserts that all the dominoes do, in fact, fall over.

Example 1

- Consider
$$Q(n) = 0 + 1 + 2 + \cdots + n.$$
- We can look at $Q(n)$ when n is small and see if we can spot a pattern:
  - $Q(0) = 0 = \frac{0 \cdot 1}{2}$
  - $Q(1) = 1 = \frac{1 \cdot 2}{2}$
  - $Q(2) = 3 = \frac{2 \cdot 3}{2}$
  - $Q(3) = 6 = \frac{3 \cdot 4}{2}$
  - $Q(4) = 10 = \frac{4 \cdot 5}{2}$
- Conclusion:
$$Q(n) = 0 + 1 + 2 + \cdots + n = \frac{n}{2}\left(n + 1\right).$$

Example 2

- Now consider

$$R(n) = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{(n+1)\cdot(n+2)}.$$

- Again, we can look at $R(n)$ when n is small and see if we can spot a pattern:
  - $R(0) = 1/2$
  - $R(1) = 1/2 + 1/6 = 2/3$
  - $R(2) = 1/2 + 1/6 + 1/12 = 3/4$
  - $R(3) = 1/2 + 1/6 + 1/12 + 1/20 = 4/5$
- Conclusion:

$$R(n) = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{(n+1)\cdot(n+2)} = \frac{n+1}{n+2}.$$

Example 3

- Finally, consider
$$S(x) = x^2 + x + 41.$$
- Again, we can look at $S(x)$ when x is small and see if we can spot a pattern:
  - $S(0) = 41$ prime!
  - $S(1) = 43$ prime!
  - $S(2) = 47$ prime!
  - $S(3) = 53$ prime!
- Keep checking: $S(4) = 61$ (prime) $S(5) = 71$ (prime), $\ldots$, $S(10) = 151$ (prime), $\ldots$
- Conclusion: $S(x)$ is always a prime.

Conclusions?

- Can we trust these 'conclusions'?
- What if the patterns we have spotted break down at some point?
- Look back at our third example: $S(x) = x^2 + x + 41$ and consider $x = 40$
  - $S(40) = 40^2 + 40 + 41 = 1681 = 41^2$ (not prime!)
- So, our 'conclusion' in this case was incorrect.
- We have been a bit hasty in making our conclusions.
- What is needed is either a counterexample (as above) or a proof.

Proofs

- Look back at Example 1: $Q(n) = 1 + 2 + \cdots + n$.
- Sometimes we can make our conclusions certain by way of simple algebraic manipulation:

$$\begin{aligned} 1 + 2 + \cdots + n &= \frac{1}{2}\left[(1 + 2 + \cdots + n) + (1 + 2 \cdots + n)\right] \\ &= \frac{1}{2}[(n + (n-1) + \cdots + 2 + 1) \\ &\quad + (1 \ + \ 2 \ + \cdots + (n-1) + n)] \\ &= \frac{1}{2}\left[(n+1) + (n+1) + \cdots (n+1)\right] \\ &= \frac{1}{2}[n(n+1)] = \frac{n}{2}(n+1). \end{aligned}$$

Proof by induction

- However, the principle of induction makes possible a powerful method of proof which we call proof by induction.
- Consider Example 2: $R(n) = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{(n+1)\cdot(n+2)}$.
- Our hypothesis (which we mistakenly called a conclusion) was that for all natural numbers n we have $R(n) = \frac{n+1}{n+2}$.
- To prove this we will show
  1. Our hypothesis is true for $n = 0$, ie. $R(0) = 1/2$; and
  2. If our hypothesis is true for $n = k$, ie. $R(k) = \frac{k+1}{k+2}$, then our hypothesis is true for $n = k + 1$, ie. $R(k+1) = \frac{k+2}{k+3}$.
- If both these statements are true then, on the basis of the principle of induction, our hypothesis is true for any natural number n.

Proof of Ex. 2 by induction

1. $R(0) = \frac{1}{1 \cdot 2} = \frac{1}{2}$;
2. Suppose our hypothesis is true for $n = k$, ie. $R(k) = \frac{k+1}{k+2}$ where k is some natural number.
   - Note that $R(k+1) = R(k) + \frac{1}{(k+2)(k+3)}$.
   - So,
     $R(k+1) = \frac{k+1}{k+2} + \frac{1}{(k+2)(k+3)} = \frac{k^2+4k+4}{(k+2)(k+3)} = \frac{(k+2)^2}{(k+2)(k+3)} = \frac{k+2}{k+3}$.

- So, on the basis of the principle of induction, $R(n) = \frac{n+1}{n+2}$ for any natural number n.

The principle of induction (PI)

More formally, the principle of mathematical induction states:

- If $S \subseteq \mathbb{N}$ is a subset of the natural numbers which has the following properties
  1. $0 \in S$; and
  2. $n \in S$ implies that $n + 1 \in S$

  then $S = \mathbb{N}$, that is S contains all natural numbers.

The principle and the proof

In our proof a moment ago We used this principle as follows:

- Let T be the set $\{n \in \mathbb{N} \mid R(n) = \frac{n+1}{n+2}\}$.
  1. We first showed that $R(0) = 1/2$, ie. $0 \in T$;
  2. Then we showed that if $R(k) = \frac{k+1}{k+2}$, then $R(k+1) = \frac{k+2}{k+3}$
     - ie. if $k \in T$, then $k+1 \in T$.
  3. Thus the principle of induction says that $T = \mathbb{N}$.
     - ie. $R(n) = \frac{n+1}{n+2}$ for any natural number n.

The principle of complete induction (CI)

A variant on the principle of induction is the principle of complete induction, which states:

- If $S \subseteq \mathbb{N}$ is a subset of the natural numbers which has the following properties
  1. $0 \in S$; and
  2. $\{0, 1, \ldots, n\} \subseteq S$ implies that $n + 1 \in S$

  then $S = \mathbb{N}$.

The well-ordering principle (WOP)

A seemingly unrelated statement is the The well-ordering principle which states that

- Every non-empty subset S of $\mathbb{N}$ has a (unique) least element, ie. there is an element $a \in S$ such that $a \leq b$ for all $b \in S$.

We will finish with the surprising result that PI, CI and WOP are logically equivalent statements.

PI $\implies$ CI

- Assume the truth of PI.
- Let S be a subset of $\mathbb{N}$ such that $0 \in S$ and whenever $\{0, 1, 2, \dots, n\} \subseteq S$, we have $n + 1 \in S$.
- We want to show that $S = \mathbb{N}$.
- Consider the statement $P(n)$: the integers $0, 1, 2, \dots n$ are in S.
- Let $S' = \{n \in \mathbb{N} \mid P(n) \text{ is true } \}$.
- Then $0 \in S'$. Assume $k \in S'$, ie. $P(k)$ is true, ie. $\{0, 1, 2, \dots, k\} \subseteq S$. Then $k + 1 \in S$, so $\{0, 1, 2, \dots, k, k + 1\} \subseteq S$ and so $P(k + 1)$ is true, ie. $k + 1 \in S'$.
- So, we have $0 \in S'$ and $k \in S' \implies k + 1 \in S'$ and so, by PI, $S' = \mathbb{N}$. However, this means that $S = \mathbb{N}$.

CI $\implies$ WOP

- Assume the truth of CI.
- Let S be a non-empty subset of $\mathbb{N}$. Assume that S does not have a least element.
- Let S' be the set of natural numbers that do not belong to S.
- Suppose $0 \in S$. Now 0 is the least element of $\mathbb{N}$ and so 0 is the least element of S contradicting our earlier assumption, so $0 \notin S$, ie. $0 \in S'$.
- Now assume $\{0, 1, 2, \cdots, k\} \subseteq S'$. If $k + 1 \in S$, then it would follow that $k + 1$ is the least element of S. So, $k + 1 \notin S$ and so $k + 1 \in S'$.
- Thus, by CI, $S' = \mathbb{N}$, so $S = \emptyset$, contradicting our assumption that S is non-empty. So, S must have a least element.

WOP $\implies$ PI

- Assume the truth of WOP.
- Let S be a subset of $\mathbb{N}$ such that $0 \in S$ and $k \in S \implies k+1 \in S$.
- Let S' be the set of natural numbers that do not belong to S. Assume S' is non-empty.
- By WOP, S' has a least element a and, by definition of S', $a \notin S$. Thus $a \neq 0$ and so $a \geq 1$ which means $a-1$ is a natural number.
- Moreover, $a-1 \notin S'$ since $a-1 < a$ and a is the least element of S'.
- However, this implies $a-1 \in S$ and so $(a-1)+1 = a \in S$ which contradicts our assumption that $a \in S'$.
- So, it is not possible that S' is non-empty and so $S' = \emptyset$ and $S = \mathbb{N}$.

Conclusion

- We have shown
  1. PI $\implies$ CI;
  2. CI $\implies$ WOP; and
  3. WOP $\implies$ PI.
- Putting these all together gives PI $\Leftrightarrow$ CI $\Leftrightarrow$ WOP.
- So, the well ordering principle is logically equivalent to the principle of mathematical induction.
- Do you find this to be surprising?

Questions?