Groups & Symmetry

Robert Heffernan

1 Introduction

Group theory is one of the pillars of modern mathematics and is an exciting and vibrant area of research activity. We have only a few lectures during which we will cover the absolute basics of the theory. In order to make this more exciting and to demonstrate how interesting mathematics can be we are going to aim toward a specific goal: the classification of what are called the *wallpaper groups*.

Of course, you don't know what a group is yet. However, we will see that while numbers are mathematical objects that measure *size*, there is a sense in which groups are mathematical objects that measure *symmetry*.

A *tessellation* or *tiling* of the plane is a collection of plane figures that fills the plane with no overlaps and no gaps. Such designs can be seen in the art of M.C. Escher or in the beautiful designs of Islamic decorative art. Consider the following beautiful example of a plane-tiling from Alhambra in Spain:

A symmetry of such a pattern is, loosely speaking, a way of transforming the pattern so that the pattern looks exactly the same after the transformation. We will see that different types of symmetry will give rise to different groups. These groups are called *wallpaper groups* or *plane crystallographic groups*.

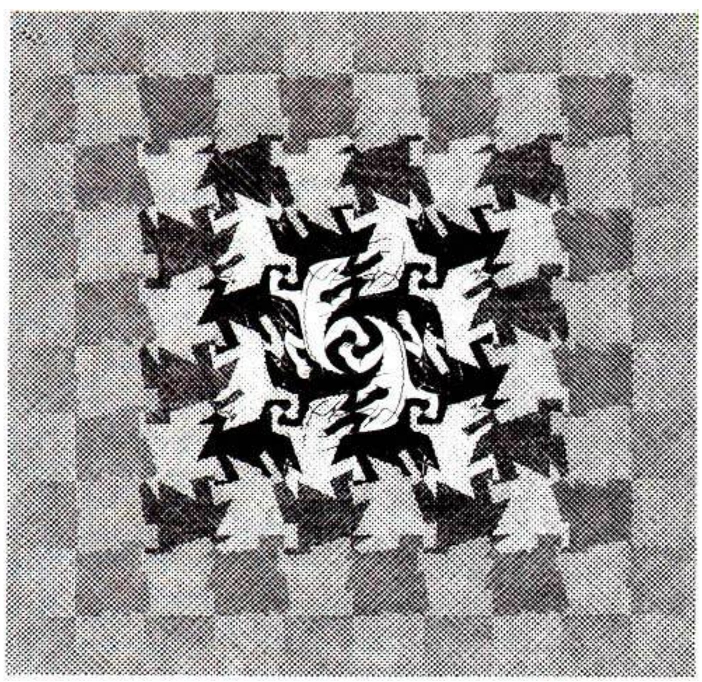
The aim of these notes is to prove the remarkable result that there are only 17 such wallpaper groups and, hence, there are essentially only 17 different tessellations of the plane that possess translational symmetry. This result was first proved by Evgraf Fedorov in 1891 and was independently proved by George Pólya in 1924.

It is sometimes said that all 17 of these patterns may be found in Alhambra although this may not be true. Certainly, Islamic art is a good place to look for examples of these symmetric tilings.

It may not be possible to finish all the material from these notes in lectures. Anything we do not cover will not be included on the examination. The slides from the lectures will be made available on WebCT.

We will make use of many of the concepts we have covered up until now in the course. In particular, you should be sure that you are comfortable with *sets* and *subsets*, *relations* (particularly *equivalence relations*), *functions* (including *injections*, *surjections* and *bijections*) and *binary operations*.

You have been given a set of problems to solve. Your solutions to these will count toward your classwork grade. These problems are designed to complement these notes and spending time at them will greatly improve your understanding of the group theory we cover as well as giving you a chance to practice proving things. If you wait until the last minute to attempt these problems you will not derive very much benefit from them (both in terms of knowledge gained and marks earned).



— *Development 1* (M.C. Escher, 1937)

2 Lecture 1

2.1 Abstract groups

We are already familiar with sets and with binary operations so we we will begin by giving the abstract definition of a group. We will then give some simple examples of groups.

Definition 2.1. Let G be a nonempty set together with a binary operation $*$. We say that G is a *group* under this binary operation if the following three properties are satisfied:

(Associativity) $(a * b) * c = a * (b * c)$ for all $a, b, c \in G$;

(Identity) There is an element $e \in G$ such that $a * e = e * a = a$ for all $a \in G$; and

(Inverses) For each $a \in G$ there is an element b in G such that $a * b = b * a = e$.

So, to specify a group one needs two things:

1. A set G (this set can be finite or infinite); and
2. An associative binary operation $*$ on G that has an identity element e and where every element has an inverse.

In our discussion of binary operations we proved the following three theorems:

Theorem 2.1. *A group G has exactly one identity element.*

Theorem 2.2. *Let G be a group and let a be an element of G. Then a has a unique inverse which we denote by a^{-1}.*

Theorem 2.3. *In a group G the* cancellation laws *hold, that is*

- *$b * a = c * a$ implies $b = c$; and*
- *$a * b = a * c$ implies $b = c$*

for any $a, b, c \in G$.

The following Lemma is also easy to prove (and useful)

Lemma 2.1. *Let G be a group with binary operation $*$.*

1. If $a \in G$, then $(a^{-1})^{-1} = a$.

2. If $a_1, a_2, \ldots, a_n \in G$, then $(a_1 * a_2 * \cdots a_n)^{-1} = a_n^{-1} * a_{n-1}^{-1} * \cdots * a_2^{-1} * a_1^{-1}$.

It is common to write ab instead of $a * b$. We will often do this. Remember, however, that a given group G need not be *commutative* and so, in general, $ba \neq ab$. If the underlying set G is finite we call G (taken together with its binary operation) a *finite group*. Otherwise G is an *infinite group*. If a group G is finite we call $|G|$ the *order* of the group G. Otherwise we say G has *infinite order*.

2.1.1 Examples (and non-examples) of groups

Example 2.1. The set of integers $\mathbb{Z}$ is a group under the binary operation of addition. In this case the identity is 0 and the inverse of a is $-a$.

Example 2.2. The set $\mathbb{Q}$ or rational numbers and the set $\mathbb{R}$ of real numbers are also groups under the operation of addition.

Example 2.3. The set of integers $\mathbb{Z}$ under the binary operation of multiplication is *not* a group. (Why?) However, the set $\mathbb{Q}^+$ of positive rational numbers *is* a group under multiplication. In this case 1 is the identity element and the inverse if a is $1/a$.

Example 2.4. The set of 2×2 matrices of real numbers with nonzero determinant is a group under the operation of matrix multiplication. This set is usually denoted by $\mathrm{GL}_2(\mathbb{R})$. So

$$\mathrm{GL}_2(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a, b, c, d \in \mathbb{R}, ad - bc \neq 0 \right\}.$$

In this case the identity is the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}.$$

The set of *all* 2×2 matrices of real numbers is usually denoted by $M_2(\mathbb{R})$. This is *not* a group under matrix multiplication. Why?

Example 2.5. Consider the set $\mathbb{Z}_n = \{0, 1, \dots, n-1\}$ where $n \geq 1$. This is a group under the binary operation of addition modulo n. The identity element is 0 and for any $j \in \mathbb{Z}_n$ the inverse of j is $n - j$. This group is often referred to as the *group of integers modulo* n and is, of course, an example of a finite group.

There are many more examples of groups—the concept of a group is everywhere once you know to look for it. We will see some more examples as we go along.

2.1.2 Abelian groups

Definition 2.2. A group G is called *commutative* or *abelian* if $ab = ba$ for all $a, b \in G$.

Exercise 2.1. Which of the examples in §2.1.1 are abelian?

2.1.3 Subgroups

Definition 2.3. A non-empty subset H of a group G is said to be a *subgroup* of G if H is a group under the group operation in G. If H is a subgroup of G, then we write $H \leq G$.

Remember that if H is a subset of the set G we write $H \subseteq G$. Such a subset is not necessarily a subgroup so be careful not to confuse the two notations. If H is a proper subset of the group G, ie. $H \subset G$ and H is a subgroup of G, then we write $H < G$ and call H a *proper subgroup* of G.

We must be careful when checking to see that a subset H is a subgroup of G. Suppose G is a group with binary operation $*$. Then $*$ is a function from $G \times G$ to G. If H is a subset of G then, for H to be a subgroup, we must have that the restriction of $*$ to $H \times H$ is a function from $H \times H$ to H. In other words, if H is a subset of G and $h_1, h_2 \in H$ then in order for H to be a subgroup of G we must have $h_1 * h_2 \in H$. In this case we say that H is *closed* under the binary operation $*$.

It is easy to show that

Theorem 2.4. *If H is a subgroup of the group G, then the identity element of G is the same as that of H, and the inverse of each element of H is the same in G as in H.*

The next theorem gives a way to check if a subset H of a group G is a subgroup:

Theorem 2.5. *A non-empty subset H of a group G is a subgroup if and only if $xy^{-1} \in H$ for every $x, y \in H$.*

Proof. It is clear that if H is a subgroup of G containing x and y, then y^{-1} is in H and so xy^{-1} is in H.

Suppose conversely that H is a nonempty subset containing xy^{-1} whenever it contains x and y. Then, if $x \in H$ we have $xx^{-1} = e$ in H and so if $y \in H$, then $ey^{-1} = y^{-1} \in H$. Also, for each $x, y \in H$, $x(y^{-1})^{-1} = xy$ is in H and so H is closed under the binary operation of G. Finally, since the associative law holds in G it also holds in H.

So, we have shown that the identity element e is in H, if y is in H then y^{-1} is in H, H is closed under the binary operation of G and the associative law holds in H. Thus H is a group under the binary operation of G and is, as such, a subgroup of G. $\square$

For any group G we have

- $G \leq G$; and
- $\{e\} \leq G$.

2.1.4 Cyclic groups

Let G be a group. Given $a \in G$ we write a^1 for a and a^2 for aa. Inductively, for $n \in \mathbb{N}$, we define a^{n+1} to be $(a^n)a$. Moreover, we define a^0 to be e and for $n \in \mathbb{N}$ we define a^{-n} to be $(a^{-1})^n$. It is easy to verify that $a^m a^n = a^{m+n}$ and $(a^m)^n = a^{mn}$.

Definition 2.4. A group G is called *cyclic* if there is an element $a \in G$ such that

$$G = \{a^n \mid n \in \mathbb{Z}\}.$$

Such an element a is called a *generator* of G. If G is a cyclic group generated by a we write $G = \langle a \rangle$.

Example 2.6. The set of integers $\mathbb{Z}$ under ordinary addition is cyclic. Both 1 and -1 are generators.

Example 2.7. The set $\mathbb{Z}_n$ under addition modulo n is cyclic. Again, 1 and $-1 = n - 1$ are generators. However, $\mathbb{Z}_n$ may have more generators than these.

3 Lecture 2 (Some more abstract group theory)

3.1 Generators

We can extend the notion of a cyclic group to talk about groups generated by more than one element:

Definition 3.1. Let G be a group and S a subset of G. The *subgroup generated by* S, denoted by $\langle S \rangle$ is the smallest subgroup of G containing S.

If $S = \{s_1, s_2, \ldots, s_n\}$ then $\langle S \rangle$ is the set of all products of powers of $s_1, s_2, \ldots, s_n$, ie.

$$\langle S \rangle = \{s_1^{\alpha_1} s_2^{\alpha_2} \cdots s_n^{\alpha_n} \mid \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{Z}\}.$$

Another way to think of this is that $\langle S \rangle$ is the intersection of all the subgroups of G which contain S.[1]

Definition 3.2. If G is a group and S is a set of elements of G such that $\langle S \rangle = G$, then we say that G is *generated by* S.

So a cyclic group is a group that is generated by one element. A group may have many different generating sets.[2]

3.2 Homomorphisms and isomorphisms

Definition 3.3. A function f from a group G into a group H is said to be a *homomorphism* if, for all $a, b \in G$, $f(ab) = f(a)f(b)$.

Notice that on the left-hand-side of this relation, ie. in the term $f(ab)$, the product ab is computed in G whereas on the right-hand-side of the relation, ie. in the term $f(a)f(b)$, the product is that of elements in H. So, a homomorphism is (loosely speaking) a function between groups that respects the binary operations of these groups.

Example 3.1. If G and H are groups and e_H is the identity element of H, then the function $f : G \to H$ given by $f(g) = e_H$ for all $x \in G$ is trivially a homomorphism.

Example 3.2. Let G be a group. The function $g : G \to H$ defined by $g(x) = x$ for all $x \in G$ is a homomorphism from G to itself.

Example 3.3. Let $\mathbb{R}$ be the group of all real numbers under the binary operation of addition and let $\mathbb{R}^*$ be the group of nonzero real numbers under the binary operation of multiplication. Define $f : \mathbb{R} \to \mathbb{R}^*$ by $f(a) = 2^a$ for each $a \in \mathbb{R}$. To see that this is a homomorphism we must check that $f(ab) = f(a)f(b)$, ie. that $2^{a+b} = 2^a 2^b$ which is, as we know, true.

Definition 3.4. A homomorphism f from G to H is said to be an *isomorphism* if f is bijective.

[1] One of the problems in your homework is to prove that the intersection of two subgroups is a subgroup. This can easily be extended to say that the intersection of finitely many subgroups is a subgroup.

[2] Another homework problem asks you to show that a finite cyclic group of order n has $\phi(n)$ generators.

Definition 3.5. Two groups G and H are said to be *isomorphic* if there exists an isomorphism between G and H. In this case we write $G \cong H$.

Let's think a minute about what it means for two groups G and H to be isomorphic. If $G \cong H$, then there exists some isomorphism $f : G \to H$. Since f is a bijective function, the sets G and H have the same cardinality, so $|G| = |H|$. The isomorphism f puts each element g of G into one-to-one correspondence with some element $f(g)$ of H and if $g = g_1 g_2 \in G$ then $f(g) = f(g_1 g_2) = f(g_1)f(g_2)$. So, the sets G and H are, at least for group-theoretical purposes, basically the same: they have the same cardinality. Moreover, the binary operations on G and H respectively are also the same.

The point is this: when two groups are isomorphic, then they are, in some sense, equal. The only difference is that their elements are labelled differently. An isomorphism f gives us a way of matching the different labellings. In fact:

Theorem 3.1. *The binary relation $\cong$ is an equivalence relation on the set of all groups:*

- $G \cong G$*;*
- $G \cong H$ *implies* $H \cong G$*; and*
- $G \cong H$ *and* $H \cong J$ *implies* $G \cong J$*.*

Proof. These follow from the (easily verified) facts that

- $f : G \to G$ given by $f(x) = x$ for all $x \in G$ is an isomorphism;
- If $f : G \to H$ is an isomorphism, then f^{-1} is an isomorphism; and
- If $f : G \to H$ and $g : H \to J$ are isomorphisms, then $g \circ f : G \to J$ is an isomorphism.

□

The binary relation of isomorphism gives rise to equivalence classes which we call *isomorphism classes*. From the point of view of abstract group theory two groups in the same isomorphism class are identical. However, for practical purposes we may prefer one or the other in a given situation.

3.3 Normal subgroups

The following definition is of great importance in group theory (although we will not make much use of it):

Definition 3.6. A subgroup N of a group G is said to be a *normal* subgroup of G if whenever $g \in G$ and $n \in N$, then $g^{-1}ng \in N$. We write $N \trianglelefteq G$.

If H is a subgroup of G we often write $g^{-1}Hg$ to mean the set of all elements of the form $g^{-1}hg$ where $h \in H$, ie.

$$g^{-1}Hg = \{g^{-1}hg \mid h \in H\}.$$

So, a normal subgroup of G is a subgroup N such that $g^{-1}Ng \subseteq N$ for all $g \in G$. In fact:

Theorem 3.2. *If $N \trianglelefteq G$, then $g^{-1}Ng = N$.*

Proof. If $g^{-1}Ng = N$ for all $g \in G$ then N is clearly normal.

Conversely, suppose $N \trianglelefteq G$. Then if $g \in G$, $g^{-1}Ng \subseteq N$ and $(g^{-1})^{-1}Ng^{-1} \subseteq N$, ie. $gNg^{-1} \subseteq N$. Now, since $gNg^{-1} \subseteq N$, $N = g^{-1}(gNg^{-1})g \subseteq g^{-1}Ng \subseteq N$ and so $N = g^{-1}Ng$. □

If you take a further course on algebra you will use the idea of a normal subgroup to define something called the *quotient group*. This is very important, but would take us further into group theory than we presently want to go. For now we note the following:

Theorem 3.3. *Let f is a group homomorphism from G to H and let e_H be the identity of H. If K is the set*

$$K = \{x \in G \mid f(x) = e_H\}$$

then K is a normal subgroup of G.

First we will prove a lemma:

Lemma 3.1. *If f is a homomorphism from G to H and e_G and e_H are the identity elements of G and H respectively, then*

1. *$f(e_G) = e_H$; and*
2. *$f(x^{-1}) = f(x)^{-1}$.*

Proof.
1. $f(x)e_H = f(x) = f(xe_G) = f(x)f(e_G)$, so $f(e_G) = e_H$.
2. $e_H = f(e_G) = f(xx^{-1}) = f(x)f(x^{-1})$ so $f(x^{-1}) = f(x)^{-1}$.

□

Proof of Theorem 3.3. First we must check that K is a subgroup of G, then we will check that it is normal.

If $x, y \in K$, then $f(x) = e_H$ and $f(y) = e_H$ and so $f(xy) = f(x)f(y) = e_He_H = e_H$ and so $xy \in K$, ie. K is closed under the binary operation of G. Also, if $x \in K$ then $f(x) = e_H$ and so, by Lemma 3.1, $f(x^{-1}) = f(x)^{-1} = e_H^{-1} = e_H$ and so $x^{-1} \in K$. Thus K is a subgroup of G.

Finally suppose $k \in K$. Then $f(g^{-1}kg) = f(g^{-1})f(k)f(g) = f(g)^{-1}e_Hf(g) = e_H$ and so $g^{-1}kg \in K$. Thus $K \trianglelefteq G$.

□

The set K is called the *kernel* of f. It should be clear that if $K = \{1\}$, then f is an isomorphism.

So, each homomorphism from G to a group H gives rise to a normal subgroup of G. It is also possible to show that each normal subgroup of G gives rise to homomorphism from G to some group H but this would again take us too far afield.

4 Lecture 3 (Some geometry)

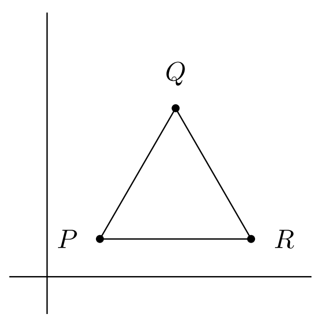
In this lecture we will introduce some of the geometry we need to talk about wallpaper groups (the formal definition of a wallpaper group will come later, when we're ready for it).

Since we are interested in group theory, and not geometry *per se*, we will state many of these results without proof.[3] It might help your geometric intuition to think a little about the results below and convince yourself that they are true. I recommend that you go home and draw some pictures to convince yourself that each theorem is true (you don't need to go so far as to write down a formal proof).

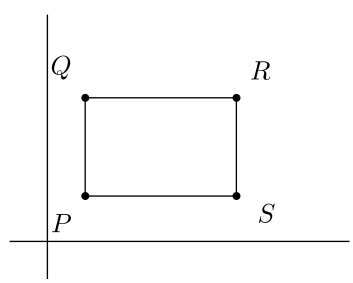
We will identify the *Euclidian plane* with the *Cartesian plane* in the usual way. So, points in the plane will be represented by ordered pairs of numbers $(x, y) \in \mathbb{R} \times \mathbb{R}$. We will denote points in the plane by capital letters $P, Q, \ldots$ and lines in the plane by lower-case letters $l, m, \ldots$. The line segment joining a point P to a point Q will be denoted by PQ and the infinite line containing the points P and Q will be denoted by $\overline{PQ}$. The *distance* from $P = (x_1, y_1)$ to $Q = (x_2, y_2)$ is defined to be $d(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_2)^2}$. The *origin* is the point $0 = (0, 0)$.

If a set S of points are all on some line we say that they are *collinear*, otherwise we say that they are *non-collinear*.

Given points P, Q, R and S we will sometimes refer to *triangles*, eg. the triangle PQR:

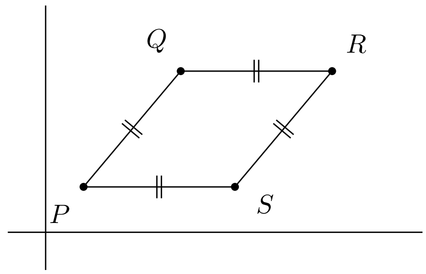


and *parallelograms* eg. the rectangle $PQRS$:



A *rhombus* is a quadrilateral whose four sides all have the same length, eg. the rhombus $PQRS$:

[3] I will also say things like 'it is clear that' and 'it is easy to show'. This is another way of saying 'I will not bother proving...'.



So, a rhombus is a special type of parallelogram and a rhombus with right angles is a square.

4.1 Transformations

Definition 4.1. A *transformation* is is a bijective function from the set of points in the plane.

If f is a transformation such that, whenever l is a line, $f(l)$ is also a line then we call f a *collineation.*

The identity transformation $\iota : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ sends P to itself for every point P, ie. $\iota(P) = P$ for all P. Note that since transformations are bijective they have inverses and it is easy to see that the inverse of a transformation is also a transformation. Also, if α and β are transformations then the composition $\beta \circ \alpha$ is also a transformation. Moreover, the composition of transformations is associative. So, the set S of transformations contains an identity element, is closed under composition and is closed under the taking of inverses. Thus, the set of all transformations of the plane forms a group.

It is also easy to see that the set of all collineations forms a group.

We will usually write $\beta\alpha$ for $\beta\circ\alpha$ and talk about the *product* (rather than the composition) of two transformations α and β.

4.2 Translations

Definition 4.2. A *translation* is a transformation of the form $(x, y) \mapsto (x + a, y + b)$ for some fixed $a, b \in \mathbb{R}$.

Theorem 4.1. *There is a unique translation taking any point P to any other point Q.*

Proof. If $P = (c, d)$ and $Q = (e, f)$ then $(x, y) \mapsto (x + (e - c), y + (f - d))$ is a translation taking P to Q. It is easy to show that this is the only translation that will do. □

The unique translation taking P to Q is denoted by $\tau_{P,Q}$.

Theorem 4.2. *Each translation is a collineation.*

Proof. Suppose that the line l has equation $ax+by+c = 0$ and $\tau_{P,Q} = (x, y) \mapsto (x+h, y+k)$. Then $\tau_{P,Q} = \tau_{0,R}$ where $R = (h, k)$ and the line $\overline{PQ}$ is parallel to the line $\overline{0R}$. So, $\tau_{P,Q}(l)$ is the line m with equation $ax + by + (c - ab - bk) = 0$. □

Theorem 4.3. *The set of all translations forms an abelian group called the* translation *group.*

4.3 Fixed points and fixed lines

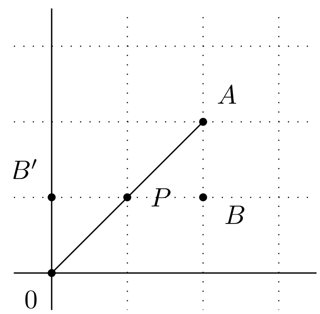
Definition 4.3. A transformation α *fixes* a point P is $\alpha(P) = P$ and fixes a line l if $\alpha(l) = l$.

Clearly, if $P \neq Q$, the translation $\tau_{P,Q}$ fixes no points and fixes exactly those lines parallel to $\overline{PQ}$.

4.4 Halfturns

Definition 4.4. If $P = (a, b)$, then the *halfturn* σ_P *about* P is the transformation $(x, y) \mapsto (-x + 2a, -y + 2b)$.

In the following illustration $P = (1, 1)$ and the halfturn σ_P takes $0 = (0, 0)$ to $A = (2, 2)$ and $B = (2, 1)$ to $B' = (0, 1)$:



Definition 4.5. A transformation α is called *involutary* if $\alpha^2 = \iota$. We also call α an *involution*. An involution is its own inverse.

The halfturn σ_P is an involutory transformation that fixes exactly the point P and fixes any line l that contains P.

Theorem 4.4. *If Q is the midpoint of PR, then $\sigma_Q \sigma_P = \tau_{P,R} = \sigma_R \sigma_Q$, ie. the product of two halfturns is a translation.*

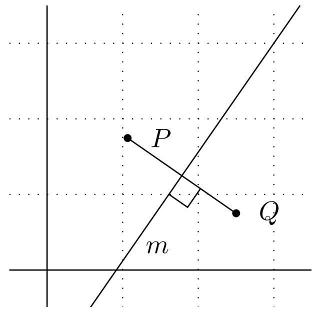
Theorem 4.5. *A product of three halfturns is a halfturn. In particular, if P,Q and R are non-collinear, then $\sigma_R \sigma_P \sigma_Q = \sigma_S$ where $PQRS$ is a parallelogram.*

Theorem 4.6. *The union of the set of translations and the set of halfturns forms a group.*

4.5 Reflections

Definition 4.6. A *reflection σ_m in the line m* is a transformation defined by

$$\sigma_m(P) = \begin{cases} \text{P if P is on m;} \\ \text{Q if P is off m and m is the perpendicular bisector of PQ.} \end{cases}$$



Reflections are obviously involutions and σ_m fixes every point on m (we say that σ_m fixes the line m *pointwise*) and fixes every line perpendicular to m (but not pointwise).

4.6 Isometries and symmetries

We will be particularly interested in a special type of transformation:

Definition 4.7. An *isometry* is a transformation α that preserves distance, ie. $d(P, Q) = d(\alpha(P), \alpha(Q))$ for all points P and Q.

Definition 4.8. An isometry α is a *symmetry* for a set S of points if $\alpha(s) \in S$ for each $s \in S$.

The set of all symmetries of a set of points forms a group. If S is the set of all points, then a symmetry of S is an isometry (and vice versa). So, the set of all isometries forms a group.

4.7 Isometries as products of reflections

What sort of group does the set of all reflections generate? Since a reflection is its own inverse, every element in this group must be a product of reflections. A product of reflections is clearly an isometry. What we will see now is that every isometry is a product of reflections.

Theorem 4.7. *If an isometry fixes two points on a line, then the isometry fixes the entire line pointwise. If an isometry fixes three non-collinear points, then the isometry must be ι.*

In fact, an isometry is completely determined by its action on three non-collinear points:

Theorem 4.8. *If α and β are isometries and there exist non-collinear points P, Q and R such that $\alpha(P) = \beta(P)$, $\alpha(Q) = \beta(Q)$ and $\alpha(R) = \beta(R)$, then $\alpha = \beta$.*

The following theorem is fundamental:

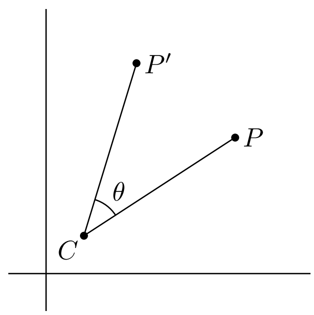
Theorem 4.9.

1. *An isometry that fixes two points is a reflection or the identity.*
2. *An isometry that fixes exactly one point is a product of two reflections.*
3. *A product of reflections is an isometry. Every isometry is a product of at most three reflections.*

4.8 Rotations

We will think of angles as being *directed*, ie. θ is different from $-\theta$. The positive direction will be anti-clockwise.

Definition 4.9. A *rotation* about a point C through a (directed) angle θ is the transformation $\rho_{C,\theta}$ that fixes C and otherwise sense P to P' where $d(C,P) = d(C,P')$ and the directed angle between $\overline{CP}$ and $\overline{CP'}$ is θ. The point C is called the *centre* of rotation.

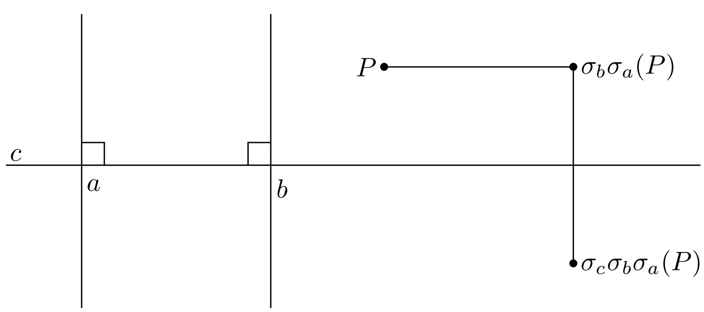


Theorem 4.10. *A rotation is an isometry and fixes exactly one point (its centre). A rotation also fixes every circle with centre C. Moreover, $\rho_{C,\theta}\rho_{C,\psi} = \rho_{C,\theta+\psi}$ and $\rho_{C,\theta}^{-1} = \rho_{C,-\theta}$.*

Theorem 4.11. *The set of rotations with centre C forms an abelian group. The involutory rotations are halfturns.*

4.9 Glide reflections

Definition 4.10. If a and b are distinct lines perpendicular to a line c, then $\sigma_c\sigma_b\sigma_a$ is called a *glide reflection* with axis c.



Note that $\sigma_b\sigma_a$ is a translation, so a glide reflection is a product of a reflection and a translation.

Theorem 4.12. *A glide reflection fixes no points and fixes exactly one line (its axis)*

4.10 Even and odd isometries

Remember that an isometry is the product of at most three reflections. An isometry that is a product of an even number of reflections is called *even*. Otherwise it is called *odd*.

Theorem 4.13. *An even isometry is a product of two reflections. An odd isometry is a reflection or a product of three reflections. No isometry is both even and odd.*

Theorem 4.14. *Every translation is the product of two reflections in parallel lines and conversely.*

Theorem 4.15. *Every rotation is a product of two reflections in intersecting lines and conversely.*

Theorem 4.16. *The halfturn σ_P is the product of two reflections in any lines perpendicular at P.*

So the product of two reflections, ie. an even isometry, is either a translation or a rotation. Only the identity is both a translation and a rotation. An odd isometry is either a reflection or a glide reflection.

Theorem 4.17. *An even involutary isometry is a halfturn, an odd involutary isometry is a reflection. The set of even isometries forms a group.*

4.11 Classification of isometries of the plane

We have seen several types of isometries and have stated some theorems regarding them. The results in the last few sections allow us to say the following:

Theorem 4.18 (Classification theorem for isometries of the plane)**.** *Each non-identity isometry is exactly one of the following: a translation, a rotation, a reflection or a glide reflection.*

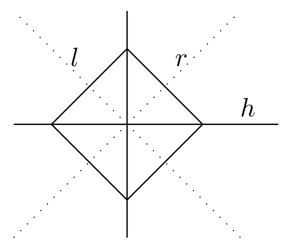
5 Lecture 4 (Finite groups of symmetries)

Recall that an *isometry* is a transformation α that preserves distance and that an isometry α is a *symmetry* for a set S of points if $\alpha(s) \in S$ for each $s \in S$.

5.1 Dihedral groups

5.1.1 The symmetry group of a square

Consider the symmetry group G of a square. For simplicity, suppose that the square is centred at the origin and that one vertex lies on the positive x-axis:



The square is fixed by $\rho = \rho_{0,90}$ and $\sigma = \sigma_h$. Note that $\sigma^2 = \iota = \rho^4$. Since G is a group, the square must be fixed by the four distinct isometries ρ, ρ^2, ρ^3 and $\rho^4 = \iota$ and the four distinct odd isometries $\rho\sigma, \rho^2\sigma, \rho^3\sigma$ and $\rho^4\sigma = \sigma$.

Now, let V_1 and V_2 be any two adjacent vertices of the square. Under any symmetry α, V_1 may go to any one of the four vertices but then V_2 must go to one of the two vertices adjacent to $\alpha(V_1)$. After this the images of the remaining vertices are determined. This means that there are at most 8 symmetries of the square and so our list above is complete. Thus the symmetry group is $G = \langle \rho, \sigma \rangle$. This symmetry group is usually denoted by D_4 and is called the *dihedral group* of order 8.

5.1.2 The symmetry group of a regular n-gon

A *regular polygon* is a polygon which is equiangular—all angles are equal in measure—and equilateral—all sides have the same length. A regular n-gon is a regular polygon with n sides. Examples include an equilateral triangle (3 sides), a square (4 sides), a pentagon (5 sides), a hexagon (6 sides), etc.

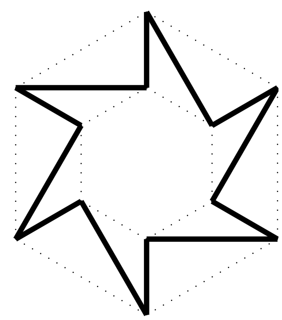
Now consider the symmetry group of a regular n-gon. Suppose again that the n-gon is centered at the origin and that one vertex lies on the positive x-axis. Keep our example of the square in mind as we proceed.

The n-gon is fixed by $\rho = \rho_{0,360/n}$ and also by reflection in the x-axis which we will denote by σ. Again, the n-gon is fixed by the n distinct even isometries $\rho, \rho^2, \ldots, \rho^n = \iota$ and by the n distinct odd isometries $\rho\sigma, \rho^2\sigma, \ldots, \rho^n\sigma = \sigma$. So the symmetry group has order at least $2n$. Let V_1 and V_2 be adjacent vertices of the n-gon. If $\alpha \in G$, then V_1 can be mapped to any other vertex by α but then $\alpha(V_2)$ must be one of the two vertices adjacent to $\alpha(V_1)$. After this the images of the other vertices are determined. Hence G has order at most $2n$ and so $|G| = 2n$. Thus, $G = \langle \rho, \sigma \rangle$. This symmetry group is called the *dihedral group* of order $2n$ and is usually denoted by D_n.

Note that the subgroup of even isometries of D_n is a cyclic group of order n. D_1 is the symmetry group of an isosceles triangle that is not equilateral, D_2 is the symmetry group of a rectangle that is not a square and D_3 is the symmetry group of an equilateral triangle.

Similarly, C_1 contains only ι and is the symmetry group of a scalene triangle[4], C_2 contains only ι and a halfturn and is the symmetry group of a parallelogram that is not a rhombus.

We have seen that the dihedral group of order $2n$ is the symmetry group of a regular n-gon. It is easy to construct a polygon that has symmetry group C_n. Here is an example of a polygon with symmetry group C_6:



A similar construction will yield polygons with symmetry groups C_n for any n.

5.2 Finite groups of symmetries

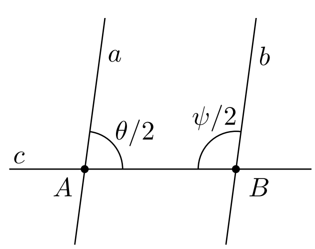
The classification of isometries of the plane tells us that an isometry is a transformation of one of four types: a translation, a rotation, a reflection or a glide reflection.

Suppose that G is a finite group of isometries. Then G cannot contain a non-identity translation or a glide reflection as each of these would generate an infinite subgroup of G. So, G contains only rotations and reflections.

Now suppose that G is a finite group of symmetries that contains only rotations. One possibility is, of course, that G is the identity group containing only ι. Let us assume that this is not the case. We will first assume that G contains at lease one non-identity rotation $\rho_{A,\theta}$. Assume that $\rho_{B,\psi}$ is another non-identity rotation in G such that $A \neq B$.

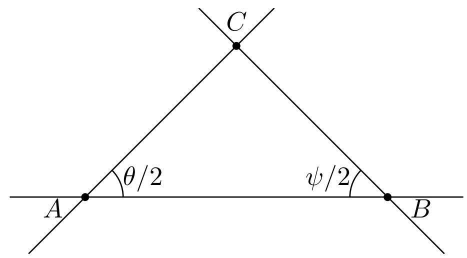
Theorem 5.1. *A rotation of θ degrees followed by a rotation of ψ degrees is a rotation of $\theta + \psi$ degrees unless $\theta + \psi = 0$, in which case the product is a translation.*

Sketch of proof. Let c be the line $\overline{AB}$. There is a line a through A and a line b through B such that $\rho_{A,\theta} = \sigma_c\sigma_a$ and $\rho_{B,\psi} = \sigma_b\sigma_c$, eg.



[4] A triangle is called *scalene* if all of its angles are different.

In the figure above a and b are parallel, ie. $\theta+\psi=0$ degrees, and so $\rho_{B,\psi}\rho_{A,\theta}$ is a translation by Theorem 4.14. The other possibility is that the lines a and b may intersect in some point C:



In this case $\rho_{B,\psi}\rho_{A,\theta}$ will be a rotation about C through an angle of $\theta+\psi$ degrees. $\square$

Back to our group G. Since G contains $\rho_{A,\theta}$ and $\rho_{B,\psi}$ it must contain

$$\rho_{B,\psi}^{-1}\rho_{A,\theta}^{-1}\rho_{B,\psi}\rho_{A,\theta}.$$

Now, by Theorem 5.1, $\rho_{B,\psi}^{-1}\rho_{A,\theta}$ is a rotation of $-(\theta+\psi)$ degrees and $\rho_{B,\psi}\rho_{A,\theta}$ is a rotation of $\theta+\psi$ degrees and, since $-(\theta+\psi)+(\theta+\psi)=0$ degrees, the product above must be a translation. If $B\neq A$, this translation will not be the identity but since we have assumed that G contains no translations, we must have $B=A$. So, all non-identity rotations in G have centre A. Since G is a group $\rho_{A,-\theta}$ is in G if and only if $\rho_{A,\theta}$ is in G. So, without loss of generality, we can write all the elements of G has $\rho_{A,\theta}$ where $0\leq\theta<360$. Now let $\rho=\rho_{A,\theta}\in G$ where θ has the minimum positive value. If $\rho_{A,\psi}\in G$ with $\psi>0$, then $\psi-k\theta$ cannot be positive and less than θ for any integer k. So, $\psi=k\theta$, ie. $\rho_{A,\psi}=\rho^k$. So, the elements of G are precisely the powers of ρ and G is a finite cyclic group.

Now assume that G is a finite group of symmetries that contains at least one reflection. Note that the set of even isometries in G form a subgroup of G. This subgroup does not contain reflections, since reflections are odd isometries, so it contains only rotations and ι. Hence, by our previous argument, this subgroup is a finite cyclic group generated by some rotation ρ. So, the even isometries are $\rho,\rho^2,\ldots,\rho^n$ for some n. Now suppose that G has m reflections in total. If σ is some reflection in G, then $\rho\sigma,\rho^2\sigma,\ldots,\rho^2\sigma$ are n odd isometries in G. This means that $n<m$. However, since the product of two reflections is either a translation or a rotation (ie. an even isometry) if we multiply each of the m reflections in G on the right by σ we will get m distinct even isometries and so $m\leq n$. Hence $m=n$ and so G contains $2n$ elements and is generated by a rotation ρ and a reflection σ. Thus G is a dihedral group D_n for some integer n.

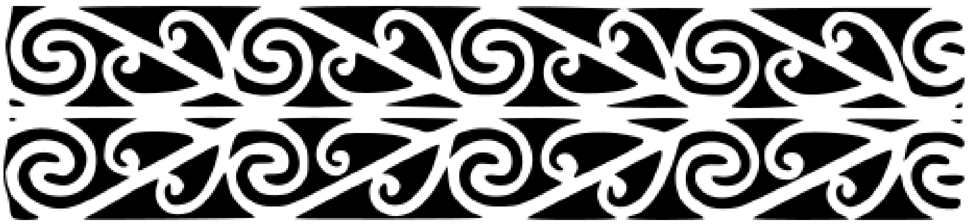
We have proved the following theorem:

Theorem 5.2 (Leonardo's theorem)**.** *A finite group of isometries is either a cyclic group C_n or a dihedral group D_n.*

6 Lecture 5 (Frieze Groups)

6.1 Introduction

In this lecture we will classify the *frieze groups*. A frieze group is a symmetry group of an infinite plane figure, such as:



whose subgroup of translations is an infinite cyclic group.

An essential property of a frieze group is this: the pattern is left fixed by some 'smallest translation'. If we write $|AB|$ for the length of the line segment AB, ie. $|AB| = d(A, B)$, then we call $|AB|$ the *length* of the translation $\tau_{A,B}$. So, what we mean by 'smallest' is 'shortest in length'.

First, some convenient terminology. Let S be a set of points. A *point of symmetry* for S is a point P such that, for each $s \in S$, $\sigma_P(s) \in S$. A *line of symmetry* for S is a line m such that, for each $s \in S$, $\sigma_m(s) \in S$.

If isometries α and σ_P are in a group G of isometries, then $\sigma_{\alpha(P)}$ is in G, since $\alpha\sigma_P\alpha^{-1}$ is in G. Similarly, if α and σ_l are in G, then $\sigma_{\alpha(l)}$ is in G. So,

Theorem 6.1. *If P is a point of symmetry for a set S of points and α is a symmetry for S, then $\alpha(P)$ is a point of symmetry for S. If l is a line of symmetry for S, then $\alpha(l)$ is a line of symmetry for S.*

Definition 6.1. A group of isometries that fix a given line c and whose translations form an infinite cyclic group is called a *frieze group* with *centre* c.

Let τ be a non-identity translation that fixes a line c. We will determine all frieze groups F with centre c whose translations form the infinite cyclic group $\langle\tau\rangle$.

We will pick a special point A on c as follows:

- If F contains halfturns, then A is chosen to be the centre of a halfturn;
- If F contains no halfturns but does contain reflections in lines perpendicular to c, then A is chosen to be the intersection of one of these lines and c;
- Otherwise, A is any point on c.

Now let $A_i = \tau^i(A)$. Then, $A_0 = A$ and, since $\tau^n(A_i) = \tau^{n+1}(A)$, each translation in F takes each A_i to some A_j.

Let M be the midpoint between A and A_1 and let $M_i = \tau^i(M)$. So, M_i is the midpoint between A_i and A_{i+1} and also the midpoint between A_0 and A_{2i+1}.

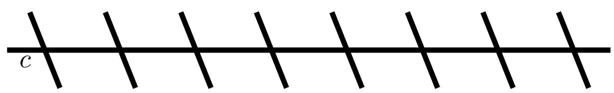
6.2 F_1

One possibility for F is that it is just the group generated by τ. Let $F_1 = \langle \tau \rangle$. A frieze pattern having F_1 as its symmetry group has no point or line of symmetry and is not fixed by a glide reflection, eg:

6.3 F_2

Apart from translations, the only other even isometries that fix c are the halfturns with centre on c. Suppose F contains a halfturn. Then $\sigma_A \in F$. Also, $\sigma_M \in F$ since $\sigma_M = \tau\sigma_A$. So, by Theorem 6.1, F contains σ_{A_i} and σ_{M_i} for each i. Now suppose P is the centre of some halfturn in F. Then $\sigma_P\sigma_A \in F$ so $\sigma_P\sigma_A(A) = A_n$ for some n, since the product of two halfturns is a translation. This implies that $\sigma_P(A) = A_n$, since σ_A fixes A, and so P is the midpoint of A and A_n, ie. $P = M_j$ where $j = (n-1)/2$. So F contains those halfturns that have centre A_i or M_i. Let $F_2 = \langle \tau, \sigma_A \rangle$. Since $\tau\sigma_n$ is an involution, $\tau\sigma_A = \sigma_A\tau^{-1}$. So, every element in F_2 is of the form τ^i or $\sigma_A\tau^i$, ie. of the form $\sigma_A^j\tau_A^i$. Also, $F_2 = \langle \sigma_A, \sigma_M \rangle$, since $\sigma_M\tau = \sigma_A$.

A freize pattern having F_2 as its group of symmetries has a point of symmetry but no line of symmetry, eg:

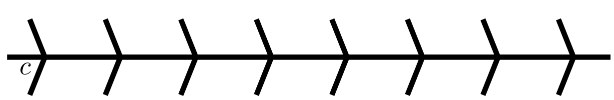


6.4 F_1^1

If F contains only even isometries it must be either F_1 or F_2. We will now try augmenting F_1 or F_2 with odd isometries.

Recall that σ_l fixes c if and only if $l = c$. Let $F_1^1 = \langle \tau, \sigma_c \rangle$. Since $\tau\sigma_c = \sigma_c\tau$, F_1^1 is abelian and every element is of the form $\sigma_c^j\tau^i$. If $n \neq 0$, then F_1^1 contains the glide reflection with axis c that takes A to A_n.

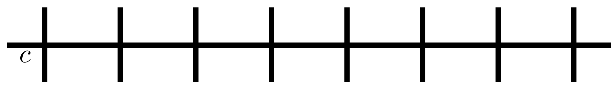
A frieze pattern with F_1^1 as its symmetry group has no point of symmetry and the centre is a line of symmetry, eg.:



6.5 F_2^1

Let $F_2^1 = \langle \tau, \sigma_A, \sigma_c \rangle$. Since σ_c commutes with τ and σ_A, every element in F_2^1 is of the form $\sigma_c^k\sigma_A^j\tau^i$. If $n \neq 0$, then F_2^1 contains the glide reflection $\sigma_c\tau^n$ with axis c that takes A to A_n. Also, F_2^1 contains $\tau^{2i}\sigma_a\sigma_c$ which is the reflection in the line perpendicular to c at M_i. If a is the line perpendiciular to c t A, then $F_2^1 = \langle \tau, \sigma_a, \sigma_c \rangle$.

A frieze pattern with F_2^1 as its symmetry group has a point of symmetry and the centre is a line of symmetry, eg.:



6.6 F_1^2

Suppose F does not contain a halfturn but does contain the reflection in a line a perpendicular to c. So, A is in a. Then F contains $\tau^{2i}\sigma_a$ which is the reflection in the line perpendicular to c at A_i and F also contains $\tau^{2i+1}\sigma_a$ which is the reflection in the line perpendicular to c at M_i.

Assume that F contains another reflection σ_l. Then $l \neq c$ as the halfturn $\sigma_c\sigma_a \notin F$. So, l is perpendicular to c. Then F contains the translation $\sigma_l\sigma_a$ which must take A to A_n for some n. So, $\sigma_l(A) = A_n$ for some $n \neq 0$ and l is perpendicular to c at some A_i or M_i. Thus F contains exactly those reflections in lines perpendicular to c and A_i for each i and M_i for each i.

We have now considered all possibilities for adding reflections to F_1.

Let $F_1^2 = \langle \tau, \sigma_a \rangle$ where a is perpendicular to c at A. Since $\tau\sigma_a = \sigma_a\tau^{-1}$, every element of F_1^2 is of the form $\sigma_a^j\tau^i$. F_1^2 does not contain σ_c but does contain the reflections in the lines perpendicular to c at A_i or M_i for each i.

A frieze pattern having F_1^2 as its symmetry group has no point of symmetry and has a line of symmetry but the centre is not a line of symmetry, eg.:

6.7 F_2^2

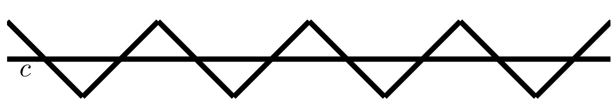
Suppose now that F contains a halfturn and also contains σ_q for some line q. If $q \neq c$, then q is perpendicular to c at A_i or q is perpendicular to c and M_i (for some i) and the group is just F_2^1. So, suppose q is off each A_i and M_i.

Since, by Theorem 6.1, $\sigma_q(A)$ must be the centre of a halfturn in F, q must be the perpendicular bisector of AM_i for some i. Theorem 6.1 now implies that F contains the reflection in the perpendicular bisector of AM_i for each i. Thus F contains σ_p where p is the perpendicular bisector of AM.

Now, if the line a is perpendicular to c at A, then F cannot contain both σ_p and σ_a as $\sigma_p\sigma_a$ is a translation taking A to M and is shorter than τ, contradicting the fact that τ is the shortest translation in F. Also, since $\sigma_p\sigma_a = \sigma_p\sigma_c\sigma_A$, F cannot contain both σ_p and σ_c. These are all the possibilities for adding a reflection to F_2.

So, let $F_2^2 = \langle \tau, \sigma_A, \sigma_p \rangle$ where p is the perpendicular bisector of AM. F_2^2 contains the glide reflection $\sigma_p\sigma_A$ with axis c that takes A to M. Let $\gamma = \sigma_p\sigma_A$. Since $\tau = \gamma^2$ and $\sigma_p = \gamma\sigma_A$ we have that $F_2^2 = \langle \gamma, \sigma_A \rangle$. F_2^2 does not contain σ_c.

A frieze pattern having F_2^2 as its symmetry group has a point of symmetry, a line of symmetry but centre is not a line of symmetry, eg.:



6.8 F_1^3

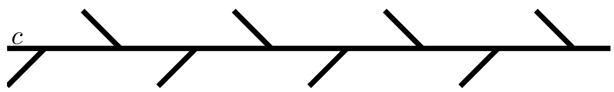
So far we have not assumed that our frieze group contains a glide reflection (although sometimes it has turned out that it does).

Now suppose F contains a glide reflection α. Then α has axis c and α^2 is a translation that fixes c. There are two cases, either $\alpha^2 = \tau^{2n}$ or $\alpha^2 = \tau^{2n+1}$ for some n.

Suppose that $\alpha^2 = \tau^{2n}$. Since α and τ commute, $(\alpha\tau^{-n})^2 = \iota$. So, the odd involuntary isometry $\alpha\tau^{-n}$ must be σ_c. Hence, $\alpha = \sigma_c\tau^n$. In this case F contains σ_c and $\sigma_c\tau^m$ for each m. If F does not contain a halfturn this is just F_1^1. If F does contain a halfturn we have F_2^1.

Now suppose that $\alpha^2 = \tau^{2n+1}$. Then $(\tau^{-n}\alpha)^2 = \tau$. Let $\gamma = \tau^{-n}\alpha$. Then γ is an odd isometry and $\gamma^2 = \tau$. Thus γ is the unique glide reflection with axis c that sends A to M. Since $\gamma^{2m} = \tau^m$ and $\gamma^{2m+1} = \tau^m\gamma$, the glide reflections in F are exactly those of the form $\tau^m\gamma$. Let $F_1^3 = \langle\gamma\rangle$.

A frieze pattern having F_1^3 as its symmetry group has no point of symmetry and no line of symmetry but is fixed by a glide reflection, eg.:



6.9 Loose ends

Finally, suppose F contains the glide reflection γ and isometries in addition to $\langle\gamma\rangle$. Since the square of the translation $\sigma_c\gamma$ is τ, $\sigma_c\gamma$ is not in $\langle\tau\rangle$. So, σ_c is not in F. If F contains σ_l with l perpendicular to c, then F also contains the halfturn $\sigma_l\gamma$. If F contains a halfturn it must contain σ_A. So, in this case, F contains σ_A and γ and so $F = F_2^2$.

We have now run out of possibilities. Thus the only possible frieze groups are F_1, F_2, F_1^1, F_2^1, F_1^2, F_2^2 and F_1^3.

6.10 Recognising frieze patterns

Each of our frieze groups is named F_i^j where i is 1 or 2 and j is 1, 2, or 3. There is method to this madness:

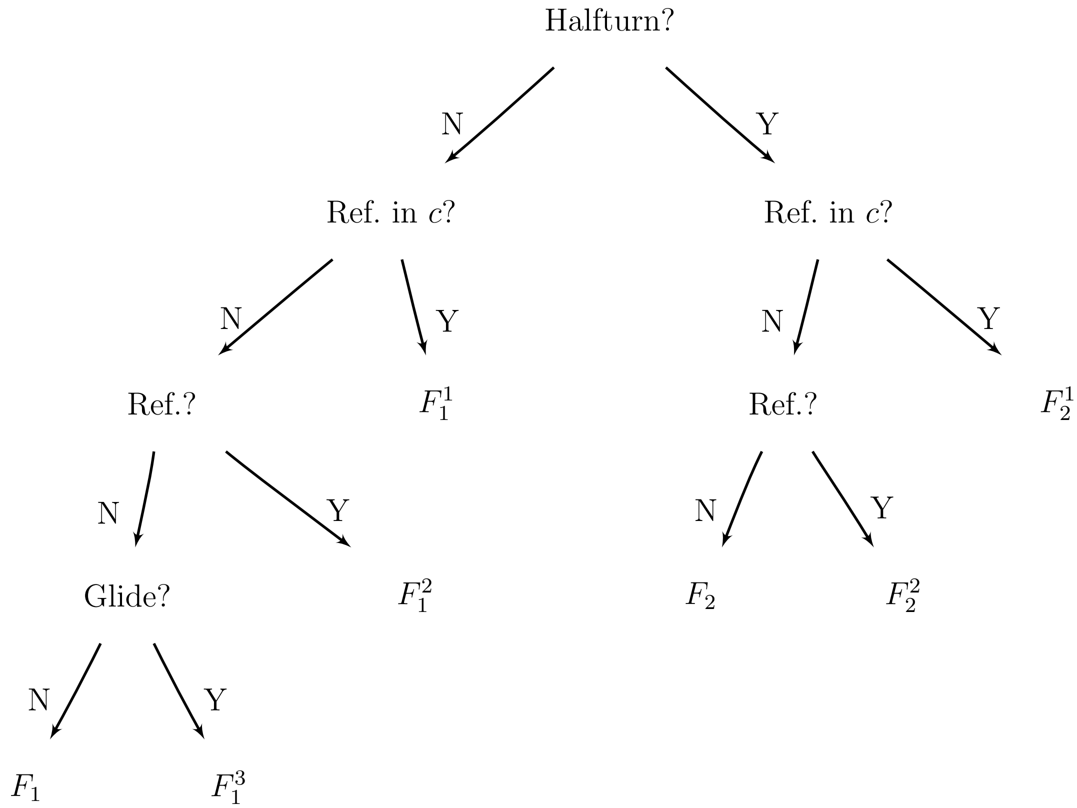
Subscript i =

- **2:** F_i^j contains a halfturn;
- **1:** F_i^j contains no halfturn.

Superscript j= **1:** F_i^j has c as a line of symmetry;

- **2:** F_i^j does not have c as a line of symmetry, but does have a line of symmetry perpendicular to c;
- **3:** F_i^j is generated by a glide reflection.

So, to recognise a frieze group you can ask yourself a series of questions:



What should I do now?

- Read back over this lecture and make sure you understand it;
- Convince yourself that we have (as we claim) covered all possibilities for frieze groups;
- Learn how to recognize the frieze group of a pattern.

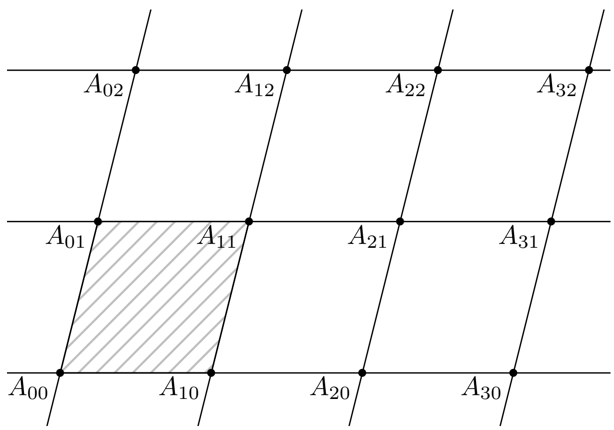
7 Lecture 6 (Wallpaper groups)

7.1 Wallpaper groups and translation lattices

Definition 7.1. A *wallpaper group* W is a group of isometries whose translations are exactly those in $\langle \tau_1, \tau_2 \rangle$ where if $\tau_1 = \tau_{A,B}$ and $\tau_2 = \tau_{A,C}$, then A, B and C are non-collinear points.

Definition 7.2. If W is a wallpaper group a *translation lattice* for W determined by a point P the set of all images of P under the translations in W.

Since every translation in a wallpaper group W is of the form $\tau_2^j \tau_1^j$, then the set of points $A_{ij} = \tau_2^j \tau_1^i(A)$ form a translation lattice for W:



A *unit cell* for W with respect to the point A and generating translation τ_1, τ_2 is a quadrilateral region with vertices $A_{ij}, A_{i+1,j}, A_{i,j+1}$ and $A_{i+1,j+1}$. An example is highlighted in the figure above.

If a translation lattice has a rectangular unit cell it is called *rectangular*, if it has a rhombic unit cell it is called *rhombic*.

We will use some preliminary results that we will not prove. The proofs of these take a little thought but are not too difficult:

Theorem 7.1. *If W contains odd isometries, then a translation lattice for W is either rectangular or rhombic.*

Theorem 7.2. *If σ_l is in a wallpaper group, then l is parallel to a diagonal of a rhombic unit cell for W or else is parallel to the side of a rectangular unit cell for W.*

Theorem 7.3. *If a glide reflection in a wallpaper group W fixes a translation lattice for W, then W contains a reflection.*

7.2 n-Centres

The previous section tells us what we need to know about odd isometries in a wallpaper group for the moment. What about rotations?

Definition 7.3. A point P is an *n-centre* for a group G of isometries if the rotations in G with centre P form a finite cyclic group C_n with $n > 1$.

A *figure* is a nonempty set of points. If P is an n-centre for the symmetry group of a figure we also call P an n-centre of the figure. A *centre of symmetry* is an n-centre for some n.

These n-centres will turn out to be pivotal[5] for our study of wallpaper groups. First notice that, for a given n, he set of n-centres must be fixed by every isometry in the group. To see this suppose that $\alpha(P) = Q$ for some isometry α in a group G. Since

$$\alpha\rho_{P,\theta}\alpha^{-1} = \rho_{Q,\pm\theta}$$

and

$$\alpha^{-1}\rho_{Q,\psi}\alpha = \rho_{P,\pm\psi}$$

we see that Q is an n-centre if and only if P is an n-centre. We have:

Theorem 7.4. *For a given n, if a point P is an n-centre for a group G of isometries and G contains an isometry that takes P to Q, then Q is an n-centre for G.*

Now suppose that rotations $\rho_{A,360/n}$ and $\rho_{P,360/n}$ with $P \neq A$ and $n > 1$ are in a wallpaper group W. Then W contains the product $\rho_{P,360/n}\rho_{A,-360/n}$ which is a non-identity translation $\tau_2^j\tau_1^i$ for some i and j by Theorem 5.1. So

$$\rho_{P,360/n} = \tau_2^j\tau_1^i\rho_{A,360/n}$$

and

$$\rho_{P,360/n}(A) = \tau_2^j\tau_1^i\rho_{A,360/n}(A) = A_{ij}.$$

Hence, either P is the midpoint of A and A_{ij} (when $n = 2$) or else the triangle APA_{ij} is isosceles. In either case

$$2|AP| = |AP| + |PA_{ij}| \geq |AA_{ij}| > 0.$$

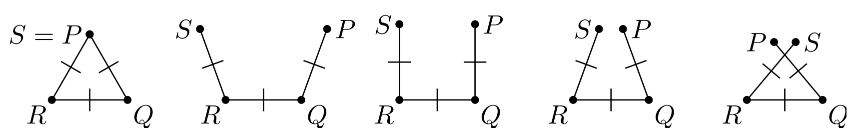
Therefore $2|AP|$ is not less than the length of any non-identity translation in W:

Theorem 7.5. *If $\rho_{A,360/n}$ and $\rho_{P,360/n}$ with $P \neq A$ and $n > 1$ are in a wallpaper group W, then $2|AP|$ is not less than the length of the shortest identity translation in W.*

This theorem tells us that no two n-centres can be 'too close' to each other.

7.3 The crystallographic restriction

Suppose a point P is an n-centre of a wallpaper group W. Let Q be an n-centre at the least possible distance from P with $Q \neq P$. Let $R = \rho_{P,360/n}(P)$. Then R is an n-centre and $|PQ| = |QR|$. Let $S = \rho_{R,360/n}(Q)$. Then S is an n-centre and $|RQ| = |RS|$.



[5] The pun is, of course intentional. If you are to read much mathematics you will need to have a strong stomach for puns.

The various possibilities are shown in the figure above. If $P = S$, then $n = 6$. If $S \neq P$, then we must have

$$|SP| \geq |PQ| = |RQ|$$

and so $n \leq 4$. We have proved:

Theorem 7.6 (The crystallographic restriction)**.** *If a point P is an n-centre for a wallpaper group, then n is one of $2, 3, 4$ or 6.*

Corollary 7.7. *If a wallpaper group contains a 4-centre, then the group contains neither a 3-centre nor a 6-centre.*

Proof. Both $\rho_{P,120}$ and $\rho_{Q,90}$ cannot be in the same wallpaper group as their product is a rotation of 30 degrees about some point Q, which the previous theorem will not allow. □

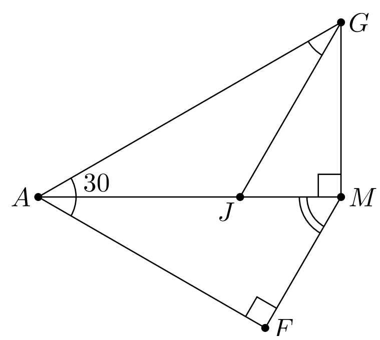
7.4 Wallpaper groups

With Theorem 7.6 in hand the idea of classifying all possible wallpaper groups becomes a much more reasonable possibility.

So that we at least see a few examples of wallpaper groups, we will look at those wallpaper groups that contain an 6-centre. We begin by noting that if we have a 6-centre for a wallpaper group, we have much more symmetry besides:

Theorem 7.8. *Suppose A is a 6-centre for a wallpaper group W. Then there are no 4-centres for W. Moreover, the centre of symmetry nearest to A is a 2-centre M, and A is the centre of a regular hexagon whose vertices are 3-centres and whose sides are bisected by 2-centres. All the centres of symmetry for W are determined by A and M.*

Proof. First, recall that since A is a 6-centre, W contains no 4-centres. Let M be an n-centre nearest to A. If M were a 3-centre or a 6-centre, then there would be a centre F closer to A than M, where $\rho_{M,120}\rho_{A,60} = \rho_{F,180}$:



and so M must be a 2-centre.

Now define a point G by the equation

$$\rho_{M,180}\rho_{A,-60} = \rho_{G,120}$$

So, G is either a 3-centre or a 6-centre. However, G cannot be a 6-centre as then there would be a centre J between A and M, where J is defined by the equation

$$\rho_{G,60}\rho_{A,60} = \rho_{J,120}.$$

Hence G must be a 3-centre.

The images of G under powers of $\rho_{A,60}$ are the vertices of the hexagon in the statement of the theorem.

Let $B = \sigma_M(A)$ and $C = \rho_{A,60}(B)$. Then B and C are 6-centres for W.

Let $N = \rho_{A,60}(M)$. Then N is a 2-centre for W.

Also, since A must go to a 6-centre under an element of W, $\sigma_M\sigma_A$ and $\sigma_N\sigma_A$ are shortest translations in W. So, $\tau_{A,B}$ and $\tau_{A,C}$ must generate the translation subgroup of W.

An illustration will help you to sort all of this out. This is on a separate sheet (which I will give you in the lecture). □

So, we have our first example of a specific wallpaper group:

$$W_6 = \langle \tau_{A,B}, \tau_{A,C}, \rho_{A,60} \rangle = \langle \rho_{A,60}, \sigma_M \rangle$$

where the triangle ABC is equilateral and M is the midpoint of AB.

In the figure (on a separate sheet) a unit cell determined by the parallelogram $ABCD$ is shaded and the midpoint of AB and is labelled M and the midpoint of AC is labelled N. E is a further point such that $NAME$ is a parallelogram.

There are two darker shaded regions in the illustration also. These regions are called *bases* for W_6. If W is a wallpaper group, a smallest polygonal region t such that the plane is covered by $\{\alpha(t) \mid \alpha \in W\}$ is called a *base* for W.

The bases help us understand how a wallpaper pattern is given to us by a symmetry group. If t' is a figure with identity symmetry group in base t, then the union of all the images $\alpha t'$ with $\alpha \in W$ is a figure with all the symmetries in W. This figure is said to have *motif* t'.

7.5 Extending W_6

What if we add more isometries to W_6 to make a new wallpaper group W? Since the rhombic translation lattice of 6-centres determined by the 6-centre A must be fixed by any isometry in W the only possibility for extending W_6 is to add reflections that fix this translation lattice. However, if we add any possible reflection the symmetries already in W_6 force us to add all possible reflections. Let

$$W_6^1 \langle \tau_{A,B}, \tau A, C, \rho_{A,60}, \sigma_{\overline{MC}} \rangle.$$

Then

$$\begin{aligned} W_6^1 &= \langle \rho_{A,60}, \sigma_M, \sigma_{\overline{MC}} \rangle \\ &= \langle \sigma_{\overline{AG}}, \sigma_{\overline{GM}}, \sigma_{\overline{MC}} \rangle \end{aligned}$$

and so W_6^1 is generated by three reflections in three lines that contain the sides of a $30-60-90$ triangle. In fact

$$W_6^1 = \langle \rho_{A,60}, \sigma_{\overline{MC}} \rangle.$$

A wallpaper pattern having W_6 as its symmetry group has a 6-centre but no line of symmetry. A wallpaper pattern W_6^1 as its symmetry group has a 6-centre and a line of symmetry.

7.6 Where do we go from here?

We have only seen two wallpaper groups but, unfortunately, we are out of time.

Theorem 7.6 is the essential tool in the classification as it restricts greatly the rotational symmetries that can appear in wallpaper groups. The classification proceeds by first considering n-centres. A wallpaper group can have no n-centre; only 2-centres or only 3-centres. A wallpaper group may also have 4-centres or 6-centres. For each of these 5 types we obtain a 'smallest' wallpaper group with each property. We then consider all possibilities of adding reflections and glide reflections to these groups. This process will yield all 17 wallpaper groups.

The appendix lists some books that you can read to see the full classification.

8 Appendix

If you enjoyed the course you may want to do more reading to learn more. The following are books that I recommend to students at your level interested in learning some more about the topics we have covered. The popular-level books are written for non-mathematicians and will make for good bedtime reading. I have chosen textbooks that I feel are both well-written and suited to your current level of knowledge. You will enjoy any of these books if you take the time to read them.

8.1 Group Theory

In my lecture notes on group theory I relied heavily on *Transformation Geometry* by George E. Martin. This book is expensive, however, and I chose it because the group-theoretical prerequisites are light. You will benefit more by reading books that take a slightly different point of view. I recommend the following books (all of which are not too expensive):

8.1.1 Popular-level books

- *Symmetry* by Herman Weyl: This book is a classic and is based on a series of lectures delivered by one of the most famous mathematicians of the last century to a general audience.

- *Symmetry* by Marcus du Sautoy: This is another well-written popular level book exploring the nature of symmetry. It is an easier read than Weyl's book.

8.1.2 Textbooks

- *Symmetries* by D.L. Johnson: If you are interested in reading the full classification of the 17 wallpaper groups (and learning more about groups and symmetry) I recommend that you read this book instead of the book by Martin mentioned above. In this book Johnson introduces more group theory than we had time for in class and he then uses this extra machinery to provide more pleasing proofs of the classifications of the frieze groups and wallpaper groups. He then moves on to tessellations of the plane and of the sphere and ends with a discussion of regular polytopes. The book is aimed at undergraduates and if you take the time to read the book you will learn some very good group theory as well as some fascinating geometry. Johnson is a very well known group theorist.

If you want to learn more group theory (and not just applications of group theory to considerations of geometrical symmetry) there are many books on the market (and many of them can be found in the university library). However, many of these books are written for graduate students and you might find them to be too difficult for a first introduction. Moreover, some of the textbooks aimed at undergraduates are not very enjoyable to read. I recommend the following books:

- *Topics in algebra* by I.N. Herstein: This a well-written and well-respected undergraduate textbook. This book contains an introduction to the theories of rings, fields and vector spaces as well as the theory of groups.

- *The theory of groups* by I.D. Macdonald: This is another well-written introduction to group theory. It is perhaps a little easier to read than Herstein's book (although both are fun to read) and concentrates entirely on group theory.

- *Topics in Group Theory* by G. Smith and O. Tabachnikova: This book is a recent publication and is again very well written and enjoyable to read.

8.2 Number Theory

8.2.1 Popular-level books

- *The music of the primes* by Marcus du Sautoy: There are many popular-level books on number theory but this is one one of the more enjoyable.

8.2.2 Textbooks

As is the case with group theory, there are many textbooks devoted to number theory but many of them are quite difficult and aimed at graduate students. If you liked the little bit of number theory we studied in the course and would like to see more, then I recommend:

- *A concise introduction to the theory of numbers* by Alan Baker: This is a very short book (only about 100 pages) and it covers the sort of elementary number theory that we looked at briefly in the course. I recommend it highly.

8.3 Set Theory and Logic

Logic is a huge area (studied by philosophers as well as by mathematicians). Again there are many textbooks on the market, many of which are quite difficult. I recommend:

- *Naive set theory* by P.R. Halmos: If you do not want to delve too deeply into the world of formal set theory but would like to learn more, then this book is ideal. P.R. Halmos is justly famous as one of the best mathematical textbook writers of his generation.

- *Sets, Logic and Categories* by Peter Cameron: As well as telling you more about set theory and logic, this book will introduce you to category theory (which is an increasingly important topic in mathematics). Cameron is another excellent writer.

- *The joy of sets* by Keith Devlin: this is a relatively difficult textbook but will be the textbook for Math499 next semester.