Directional fields

Objectives

1. **What are direction fields?**
2. **Why do we need to learn to plot direction fields?**
3. **Plotting direction fields using MATLAB**
4. **Example**
5. **Direction fields and solutions on same graph window**

1. What are direction fields?

Consider the 1st order differential equation

$$\frac{dy}{dx} = f(x,y) \qquad (*)$$

- Let $y(x)$ be a solution of $\frac{dy}{dx} = f(x,y)$ whose graph is a **solution curve** of ODE.

- If (x_0, y_0) is a point on solution curve, then the **slope of the tangent line to the solution curve** at (x_0, y_0) is given by

$$f(x_0, y_0) \qquad \{\text{since } \left.\frac{dy}{dx}\right|_{(x_0,y_0)} = f(x_0, y_0)\}$$

Key point: we can get information about slope, of the solution curve, at any point directly from Eq. (*) {without solving it}.

- Hence, **we can draw small line segments with slope $f(x_i, y_i)$ at any desired point (x_i, y_i).** *{Usually we take a reasonably good collection of points to draw these lines}.*

- **The set of all these line segments is called a direction field (or slope field);** because at each point (x_i, y_i) it gives direction (or slope) of the tangent line, to the solution curve, at (x_i, y_i).

2. Why do we need to learn to plot direction fields

- Many time it will be impossible or too difficult to solve ODE of the type given by Eq. (*).
- But by plotting direction fields, you can get a good (geometric) idea about the solution and its properties.
- It is extremely useful because
  - Plotting direction fields is quite easy (you don't have to solve ODE, you use it as it is).
  - It gives a pretty good idea of how the solution should like (because a solution curve is tangent to "small line segments" that are plotted)
  - You can use it as an aid to verify your approximate or numerical solutions.

3. Plotting direction fields in MATLAB

a) Aim

To plot direction fields of

$$\frac{dy}{dx} = f(x, y) \qquad (*)$$

b) What do we need to do

- Choose points where we want to draw slope fields
- Find the slopes at these points directly from Eq. (*).
- Draw the slope fields

c) We need two main commands of MATLAB

- **meshgrid**

  >>[x,y]=meshgrid(a:k:b,c:j:d);

  creates a set of points (x,y) where

  i. 'x' lies between 'a' & 'b', incremented by 'k'.

  ii. 'y' lies between 'c' & 'd', incremented by 'j'.

  e.g. >>[x,y]=meshgrid(1:0.5:2,0:1:2);

  creates the set of nine points

  (1,0), (1,1), (1,2),

  (1.5,0), (1.5,1), (1.5,2),

  (2,0), (2,1), (2,2).

  See explanation in class

- **quiver**

  >> quiver(a,b,x,y)

  begins at the point (a,b) and plots the vector v=(x,y).

4. Example

a) Aim

To plot direction fields of $\frac{dy}{dx} = e^{-x} - 2y$

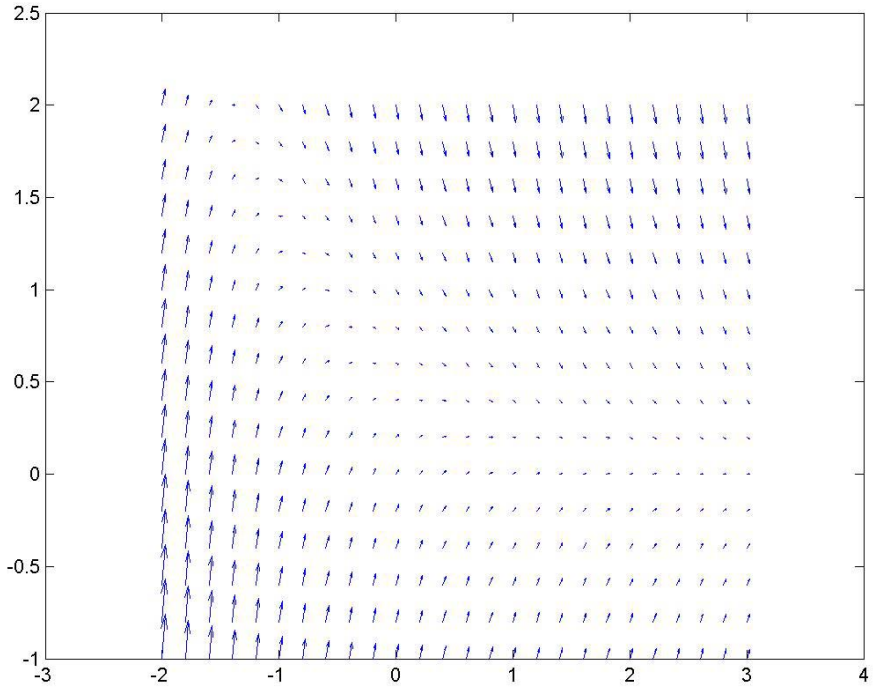
on the rectangle $-2 \le x \le 3,\ -1 \le y \le 2$.

b) Can be done using following MATLAB commands

```
>>[x,y]=meshgrid(-2:0.2:3,-1:0.2:2);
>>dy=exp(-x)-2*y;
>>dx=ones(size(dy));
>>quiver(x,y,dx,dy);
```

Why

These commands generate the following.



Note: This is really an ugly picture.

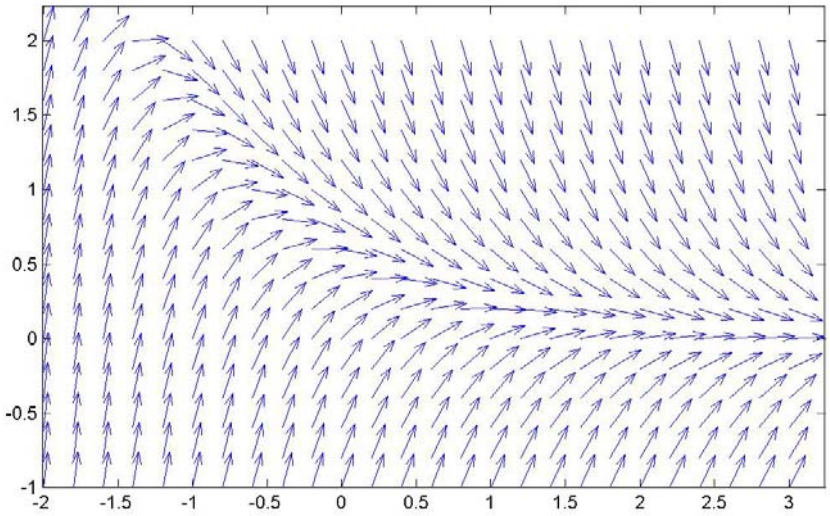
One thing we observe that arrows have different lengths. Since we are only concerned with direction, it is a good idea to make all vectors of unit length.

For this purpose, instead of before using "quiver" we continue with following commands

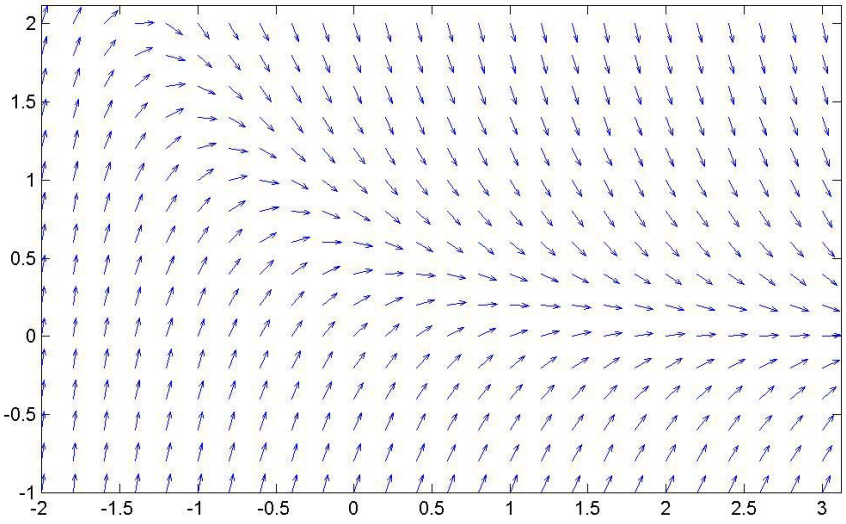
```
>>dyu=dy./sqrt(dx.^2+dy.^2);
>>dxu=dx./sqrt(dx.^2+dy.^2);
>>quiver(x,y,dxu,dyu);
>>axis equal tight
```

Why

With these commands we get the following.



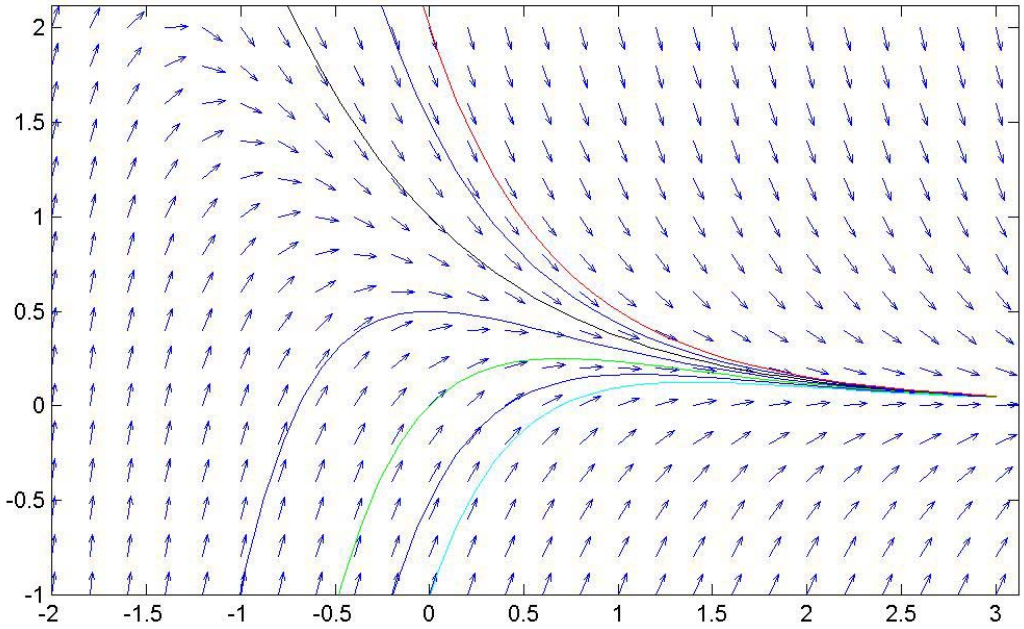
We can do better using the command **>>quiver(x,y,dxu,dyu,0.5); (why)** which gives



See file dir1.m
for all commands

5. Direction fields & solutions on same graph window

- Once you have plotted the direction field you can plot the solution curves (if you know the general solution) or your numerical solution on the same graph window using "**hold on**" and "**hold off**" as explained earlier.
- The general solution of the ODE in our example is $y = e^{-x} + Ce^{-2x}$. The following figure shows direction fields with some of these solutions.



See file dir2.m (downloadable from course page)
for all commands

End of lecture