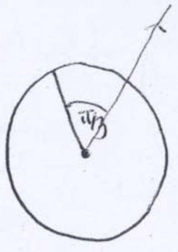
10

1. (a) Find two vectors of length 2 which make an angle of $\pi/3$ with the vector $[2, 4]$. 6

(b) Find the area of the parallelogram determined by the points 5

$P(1, 1, 1), Q(1, 2, 1), R(-1, 2, 5)$.

(a)



A vector of unit length in direction of $[2,4]$ is $\frac{1}{\sqrt{4+16}}[2,4]$

$= \frac{1}{\sqrt{5}}[1,2] = [\cos\theta, \sin\theta]$ 1

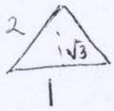
So a vector of unit length making an angle of $\pi/3$ is

$\vec{v_1} = [\cos(\theta + \pi/3), \sin(\theta + \pi/3)]$ 2

Another vector of unit length making an angle of $\pi/3$ is $\vec{v_2} = [\cos(\theta - \pi/3), \sin(\theta - \pi/3)]$.

Now $\cos(\theta + \pi/3) = (\cos\theta)\cos\pi/3 - \sin\theta \sin\pi/3$

$\sin(\theta + \pi/3) = \sin\theta \cos\pi/3 + \cos\theta \sin\pi/3$



$\cos(\theta - \pi/3) = \cos\theta \cos\pi/3 + \sin\theta \sin\pi/3$

$\sin(\theta - \pi/3) = \sin\theta \cos\pi/3 \cdot \cos\theta \sin\pi/3$

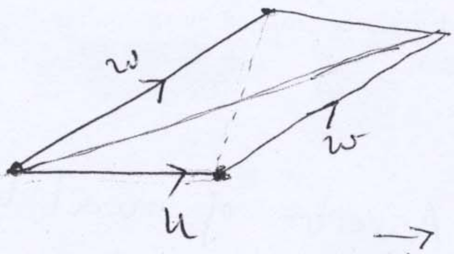
So $\vec{v_1} = \left[\frac{1}{\sqrt{5}}\frac{1}{2} - \frac{\sqrt{3}}{2}\left(\frac{2}{\sqrt{5}}\right), \frac{2}{\sqrt{5}}\frac{1}{2} + \frac{1}{\sqrt{5}}\frac{\sqrt{3}}{2}\right]$

$= \left[\frac{1}{2\sqrt{5}} - \frac{\sqrt{3}}{\sqrt{5}}, \frac{1}{\sqrt{5}} + \frac{\sqrt{3}}{2\sqrt{5}}\right]$ 1

$\vec{v_2} = \left[\frac{1}{\sqrt{5}}\frac{1}{2} + \frac{2}{\sqrt{5}}\frac{\sqrt{3}}{2}, \frac{2}{\sqrt{5}}\frac{1}{2} - \frac{1}{\sqrt{5}}\frac{\sqrt{3}}{2}\right]$

$= \left[\frac{1}{2\sqrt{5}} + \frac{\sqrt{3}}{\sqrt{5}}, \frac{1}{\sqrt{5}} - \frac{\sqrt{3}}{2\sqrt{5}}\right]$ 1

(b)



Diagonals are

$$\vec{u}+\vec{w}, \quad -\vec{u}+\vec{w}$$

If diagonals are perpendicular, then

$$(\vec{u}+\vec{w}).(-\vec{u}+\vec{w})=0$$

$$-|\vec{u}|^2+\vec{w}.\vec{w}-\vec{w}.\vec{u}+|\vec{w}|^2=0$$

So $|\vec{u}|^2=|\vec{w}|^2$

[illegible]

Conversely, if $|\vec{u}|^2=|\vec{w}|^2$, then the diagonals are perpendicular, by the same calculation.

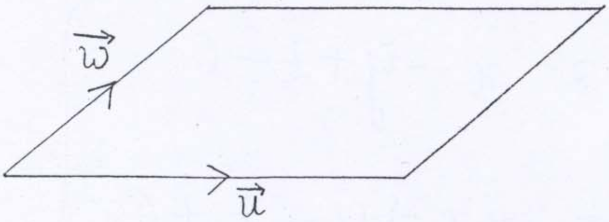
5

5 2. (a) Find components of $\vec{v}$

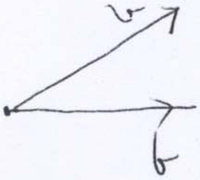
(i) along $\vec{b}$ and

(ii) perpendicular to $\vec{b}$, where $\vec{v} = [2, -1, 3]$, $b = [2, 2, 1]$.

5 (b) Use vectors to show that the diagonals of a parallelogram are of equal length if and only if the diagonals are perpendicular. $|\vec{u}| = |\vec{w}|$



(a)



Comp. of $\vec{v}$ along $\vec{b}$ = $\text{Comp}_{\vec{b}}\vec{v}$

$$= \frac{\vec{v}\cdot\vec{b}}{|\vec{b}|}\,\frac{\vec{b}}{|\vec{b}|} \quad (2)$$

Comp. of $\vec{v}$ perp. to $\vec{b}$ is:

$$\vec{v} - \text{Comp}_{\vec{b}}\vec{v} \quad (1)$$

$$\vec{v}\cdot\vec{b} = 4 - 2 + 3 = 5, \quad |\vec{b}| = \sqrt{4+4+1} = 3$$

So $\text{Comp}_{\vec{b}}\vec{v} = \left(\frac{5}{3}\right)\frac{1}{3}[2, 2, 1] = \left[\frac{10}{9}, \frac{10}{9}, \frac{5}{9}\right]$

Comp. of $\vec{v}$ perp. to $\vec{b}$ $= \left[2 - \frac{10}{9}, -1 - \frac{10}{9}, 3 - \frac{5}{9}\right]$

$$= \left[\frac{8}{9}, \frac{-19}{9}, \frac{22}{9}\right] \quad (2)$$

So $(x+y+z-6) = \pm(x-y+z-6)$

We get the planes

2 $\boxed{\begin{aligned} x+y+z-6 &= x-y+z-6 \\ x+y+z-6 &= -x+y-z+6 \end{aligned}}$

or $\boxed{\begin{aligned} 2y &= 0 \\ 2x+2z-12 &= 0 \end{aligned}}$

The normals of these planes are

$\vec{n_1} = [0,2,0]$, $\vec{n_2} = [2,0,2]$

2 Now $\vec{n_1}\cdot\vec{n_2} = 2(0)+0(2)+(0)(2) = 0$

So these planes are perpendicular

(10) pts

3. (a) Find the distanced of the point $(1,2,1)$ from the plane $x+y+z=6$.

2 (b) Find the equation of the sphere whose center is $(1,2,1)$ and which is tangent to the plane $x+y+z=6$.

2 (c) Show that all points $P(x,y,z)$ equidistant from the planes $x+y+z=6$ and $x-y+z=6$ are given by the equations

$$x+y+z-6=\pm(x-y+z-6).$$

4 (d) Use (c) to show that points equidistant from the planes $x+y+z=6$, $x-y+z=6$ lie on 2 perpendicular planes.

Soln: (a) Equation of plane in standard form is:

$x+y+z-6=0$

Dist of $P(1,2,1)$ from the plane is

2 $$\left|\frac{1+2+1-6}{\sqrt{3}}\right| = \frac{2}{\sqrt{3}}.$$

(b) The sphere centered at $(1,2,1)$ and tangent to the plane $x+y+z=6$ has radius $= \frac{2}{\sqrt{3}}$.

2 So its equation is:

$$(x-1)^2+(y-2)^2+(z-1)^2=\left(\frac{2}{\sqrt{3}}\right)^2=\frac{4}{3}.$$

(c) $P(x,y,z)$ is equidistant from the planes $x+y+z=6$ and $x-y+z=6$ iff

$$\left|\frac{x+y+z-6}{\sqrt{3}}\right| = \left|\frac{x-y+z-6}{\sqrt{3}}\right|$$ 2

(b) Direction vector of the line is

$[1, 1, -3]$

So equation of the plane is:

2 $\quad 1(x-1) + 1(y-2) - 3(z+1) = 0$

$$x + y - 3z - 6 = 0$$

Rpl6

5 4. (a) Find the line of intersection of the planes $2x + y + z = 2$ and $x + 2y + z = 3$.

2 (b) Find the equation of the plane through $(1, 2, -1)$ which is perpendicular to the line you found in part (a).

4: (a)

$$2x + y + z = 2 \quad -(1)$$
$$x + 2y + z = 3 \quad -(2).$$

From (1): $y = 2 - 2x - z \quad -(3)$

Substituting in (2),

$$x + 2(2 - 2x - z) + z = 3$$
$$-3x - z + 4 = 3$$

So $\boxed{z = -3x + 1} \quad -(4)$

From (3), $y = 2 - 2x - z$
$= 2 - 2x + 3x - 1$
$= x + 1$

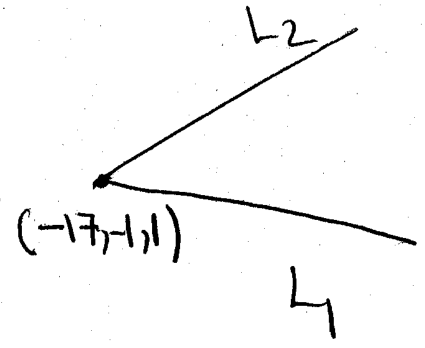
So line of intersection is given by

$$\boxed{y = x + 1, \quad z = -3x + 1}$$

Independent variable is x. Put $x = t$

So line has equations: $\boxed{\begin{array}{l} x = t \\ y = 1 + t \\ z = 1 - 3t \end{array}, \ t \in \mathbb{R}}$ 5

(b)



(1) Direction vector of L_1 is $\vec{v_1} = [4, 1, 0]$

(1) Direction vector of L_2 is $[12, 6, 3]$

$\vec{v_2} = [4, 2, 1]$

(2)
$$\vec{v_1} \times \vec{v_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 1 & 0 \\ 4 & 2 & 1 \end{vmatrix} = \vec{i}(1) - \vec{j}(4) + \vec{k}(4)$$

So equation of the required plane is:

(1)
$$(x+17)(1) + (y+1)(-4) + (z-1)(4) = 0$$

or $\boxed{x - 4y + 4z + 9 = 0}$

5. (a) Show that the lines

5

$$L_1: \quad x+1=4t, \quad y-3=t, \quad z-1=0$$
$$L_2: \quad x+13=12t, \quad y-1=6t, \quad z-2=3t$$

intersect.

5

(b) Find an equation of the plane containing the lines L_1 and L_2 (given in (a)).

Soln: (a)

$$\boxed{\begin{array}{l} x+1=4t, \quad y-3=t, \quad z=1 \\ x+13=12s, \quad y-1=6s, \quad z-2=3s \end{array}}$$

are the two lines.

2

$$\boxed{\begin{array}{l} x=4t-1, \quad y=t+3, \quad z=1 \\ x=12s-13, \quad y=6s+1, \quad z=3s+2 \end{array}}$$

So a pt. of intersection is given by

$$4t-1=12s-13, \quad t+3=6s+1, \quad 1=3s+2$$

From 3rd equation $s=-1/3$

from 2nd $t=6s-2=-4$

Plugging in 1st equation we get

$$4(-4)-1=12\left(-\frac{1}{3}\right)-13$$

3

$$-17=-17.$$

So the lines intersect at the point $(-17,-1,1)$

given by

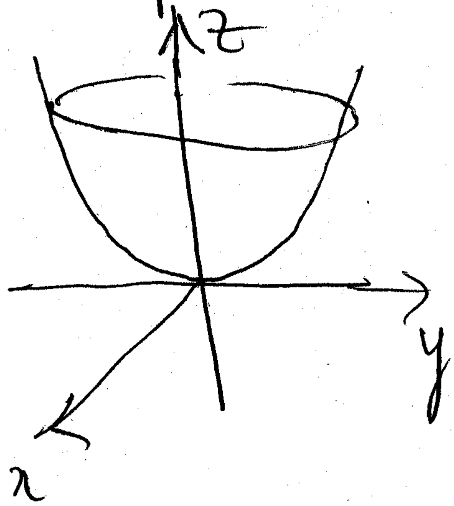
2) $\sqrt{x^2+y^2+(z-1)^2} = |z+1|$

or $x^2+y^2+(z-1)^2 = (z+1)^2$

$\therefore x^2+y^2+z^2-2z+1 = z^2+2z+1$

2 $x^2+y^2 = 4z$

which is a paraboloid opening in z direction



2

6. (a) Find the equations of the surfaces obtained by rotating the curve $y = 2z^2$ in the yz-plane about

(2)

   (i) the y-axis

   (ii) the z-axis

(8) (b) Find an equation of the surface consisting of all points $P(x, y, z)$ equidistant from the point $(0, 0, 1)$ and the plane $z = -1$. Identify and sketch this surface.

Soln: (a) (i) Rotate about y- axis : surface equation is:

(1)
$$y = 2\left(\sqrt{x^2+z^2}\right)^2 = 2(x^2+z^2)$$

(ii) Rotate about z- axis : equation is

(1) $$\sqrt{y^2+x^2} = 2z^2$$

(b) Distance of $P(x,y,z)$ from $(0,0,1)$ is

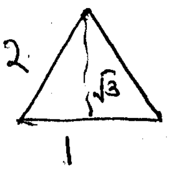
(1) $$\sqrt{x^2+y^2+(z-1)^2}$$

Dist of $P(x,y,z)$ from the plane

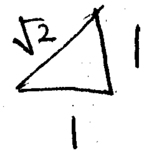
$z+1=0$ is

(1) $$|z+1|$$

So the surface equidistant from the point $(0,0,1)$ and the plane $z=-1$ is



$\theta = \pi/3$



$\varphi = \frac{\pi}{4} + \pi = 5\pi/4.$

$\therefore$ Spherical coords are

(2) $(\rho, \theta, \varphi) = (\sqrt{8}, \pi/3, 5\pi/4).$

(c) $0 \leq r \leq 2\sin\theta, \quad 0 \leq z \leq 3.$

$r = 2\sin\theta \Leftrightarrow r^2 = 2r\sin\theta$

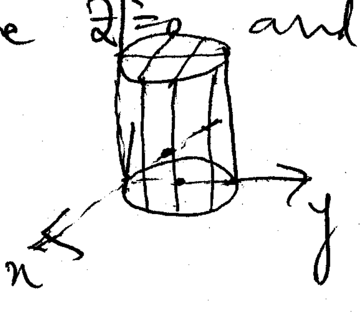
(2) $\Leftrightarrow x^2+y^2 = 2y \quad \therefore x^2+y^2-2y=0$

$x^2+(y-1)^2 = 1$

This is the cylinder centred at $(0,1)$ of radius 1.

So the inequalities describe a region inside this cylinder, bounded below by the

(3) plane $z=0$ and above by the plane $z=3$



7. (a) Find the equation of the surface $z = 3x^2 + 3y^2$ in (i) cylindrical (ii) spherical coordinates. (6)
(b) Find spherical coordinates of the point with rectangular coordinates $(1, \sqrt{3}, -2)$. (2)
(c) Describe the region in space that satisfies the inequalities (5)

$$0 \le r \le 2\sin\theta, \quad 0 \le z \le 3.$$

Sketch this region.

7(a): cylindrical coords are:

(2) (i) $x = r\cos\theta, \; y = r\sin\theta, \; z = z$

Spherical coordinates are:

(2) $x = \rho\sin\varphi\cos\theta, \; y = \rho\sin\varphi\sin\theta, \; z = \rho\cos\varphi$

Equation $z = 3x^2 + 3y^2$ in cylindrical coords is

$$\boxed{z = 3r^2}$$

In spherical coordinates it is

$$\rho\cos\varphi = 3[\rho^2\sin^2\varphi]$$

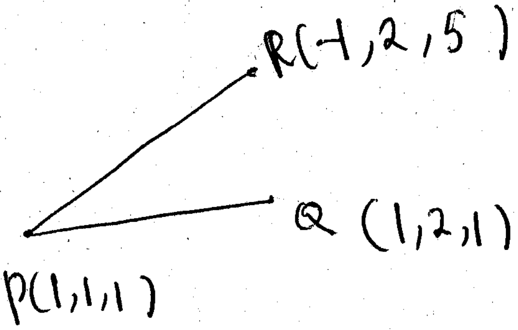
or $\boxed{\cos\varphi = 3\rho\sin^2\varphi}$

(1) $\rho = \frac{1}{3}(\cot\varphi)\csc\varphi$

(b) $\rho = \sqrt{x^2+y^2+z^2}, \quad \tan\theta = \frac{y}{x}, \quad \cos\varphi = \frac{z}{\rho}$

$\therefore \rho = \sqrt{1+3+4}, \quad \tan\theta = \sqrt{3}, \quad \cos\varphi = \frac{-2}{\sqrt{8}} = \frac{-1}{\sqrt{2}}$

(b)



$$\vec{PQ} = [0,1,0] \quad , \quad \vec{PR} = [-2,1,4]$$ 2

area $= |\vec{PQ} \times \vec{PR}| = \left\| \begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 0 \\ -2 & 1 & 4 \end{matrix} \right\|$ 2

$= |\vec{i}\,4 - \vec{j}\,0 + \vec{k}\,(2)| = \sqrt{20}$ 1