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# Quasi-potentials and Kähler–Einstein metrics on flag manifolds, II

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## Abstract

For a homogeneous space  $G/P$ , where  $P$  is a parabolic subgroup of a complex semisimple group  $G$ , an explicit Kähler–Einstein metric on it is constructed. The Einstein constant for the metric is 1. Therefore, the intersection number of the first Chern class of the holomorphic tangent bundle of  $G/P$  coincides with the volume of  $G/P$  with respect to this Kähler–Einstein metric, thus enabling us to compute volume for this metric and for all Kählerian metrics on  $G/P$  invariant under the action of a maximal compact subgroup of  $G$ .

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## 1. Introduction

Let  $G$  be a complex reductive group,  $P$  a parabolic subgroup of  $G$  and  $K$  a maximal compact subgroup of  $G$ . In [3] it is shown that the flag manifold  $G/P$  admits a  $K$ -invariant Kähler–Einstein metric whose Ricci-form coincides with the Kähler form for this metric (see Section 4 for definitions). This metric is thus an Einstein metric with Einstein constant 1 [4, p. 319]. We will refer to this metric as the canonical metric on the flag manifold  $G/P$ .

Let  $Q$  be another parabolic subgroup of  $G$  containing  $P$ . Although, in general, a Kähler–Einstein metric does not have good restriction properties, in other words, the restriction of a Kähler–Einstein to a complex submanifold need not be a Kähler–Einstein

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metric on the submanifold, the restriction of the canonical metric to any fiber of the natural projection

$$G/P \rightarrow G/Q$$

is again Kähler–Einstein: this is the main result of [3]. The arguments in [2,3] are incomplete at certain places. Namely, the correspondence between the additive characters of the parabolic group  $P$  and  $K$ -invariant Kähler forms of arbitrary signature was not proved there, and the classification of those characters of  $P$  which give rise to positive definite metrics was formulated incorrectly there. These are now formulated precisely as Propositions 2.1, 2.2 and Theorem 3.1 in this paper.

In the final section we have computed the total volume of  $G/P$  for the canonical metric (see Theorem 6.1), using essentially a result of A. Borel and F. Hirzebruch in [5]. Our motivation here was to obtain a formula analogous to the volume of  $\mathbb{C}P^n$  relative to the Fubini–Study metric on it (see [9, p. 109, Theorem 1]).

Following a suggestion of the referee, a few words about the circle of ideas considered in [3] and in this paper are in order. If  $K$  is a compact Lie group and  $L$  a subgroup of  $K$  of maximal rank then, by a classical result of H.-C. Wang, the homogeneous space  $K/L$  admits a complex structure if and only if  $L$  is the centralizer of a torus [12, Theorem C]. One can then realize  $K/L$  as a flag manifold  $G/P$ . Working with the compact group, all the non-degenerate Kähler forms are described in [13, §8]. On the other hand, the Fubini–Study metric on  $\mathbb{C}P^n$  has the potential function  $dd^c \log(1 + |z|^2)$  (see [9, p. 109]). We wanted to understand the appearance of the logarithm here from a group-theoretic viewpoint and to obtain a similar expression for  $K$ -invariant metrics on  $G/P$ . It turns out that the logarithm makes its appearance in the description of additive characters of  $P$  and, in the case of  $\mathbb{C}P^n = \mathrm{SL}(n+1, \mathbb{C})/P$ , the function  $1 + |z|^2$  arises as  $|g.v|^2$ , where  $g \in G$  and  $v$  is a highest weight vector for the natural representation of  $\mathrm{SL}(n+1, \mathbb{C})$ . Moreover, this analogy is exact in the sense that the Kähler metrics on  $G/P$ , of arbitrary signature, are generated, as described in Section 2, by highest weight vectors of certain irreducible representations of  $G$ . This means that the Kähler geometry of  $G/P$  is tied very closely to É. Cartan’s theory of highest weight vectors in a very natural manner. The reader might like to compare this with the treatment of the same problems in [4, Chapter 8].

We refer the reader to [10] and [11] for results on roots and representations used in this paper.

This paper is organized as follows. In Section 2 we recall the concept of quasi-potentials and establish its basic properties: a  $K$ -invariant function  $f$  on  $G$  is a quasi-potential if the form  $dd^c f$  can be pushed down to  $G/P$ . In Section 3 a characterization of those quasi-potentials is given which give non-degenerate forms on  $G/P$  and the cone of Kähler metrics is described in terms of these functions. For the convenience of the reader we have given, in Section 4, a treatment of Ricci forms from first principles. A Kähler form  $\omega$  defines an Einstein metric if the Ricci form of its volume form is a multiple of  $\omega$ . In Section 5, we show that there is exactly one quasi-potential which gives an Einstein metric. We also give there a more transparent proof of the main result of [3]. In the final section, we have computed the total volume of  $G/P$  relative to any  $K$ -invariant Kähler metric on  $G/P$ .

## 2. Quasi-potentials

We may assume that  $G$  is semisimple since a parabolic subgroup contains the center of  $G$ . The Lie algebra of  $G$  will be denoted by  $\mathfrak{g}$ . Fix a maximal torus  $T$  of  $G$  and a Borel subgroup  $B \supset T$ . Let  $S$  denote the set of simple roots defined by the pair  $(B, T)$ .

The intersections of a fixed parabolic subgroup with any maximal compact subgroup are isomorphic. We make a specific choice of a maximal compact subgroup  $K$  in the following way.

For any root  $\alpha \in S$ , we can choose root-vectors  $X_\alpha, X_{-\alpha} \in \mathfrak{g}$  so that  $X_\alpha, X_{-\alpha}$  and  $H_\alpha := [X_\alpha, X_{-\alpha}]$  together form a standard copy of  $\mathfrak{sl}(2, \mathbb{C})$ . Let

$$\phi_\alpha : \mathrm{SL}(2, \mathbb{C}) \rightarrow G \quad (2.1)$$

be the corresponding homomorphism. We denote by  $L_\alpha$  the image of  $\mathrm{SL}(2, \mathbb{C})$  by  $\phi_\alpha$ . The subgroup of  $G$  generated by all  $\phi_\alpha(\mathrm{SU}(2))$ ,  $\alpha \in S$ , is a maximal compact subgroup of  $G$  [11] (see the proof of [11, Lemma 45 and Corollary 1, p. 105]). We will denote this maximal compact subgroup by  $K$ .

Choose a root vector  $X_r$ , where  $r$  runs over positive roots, such that  $X_r, X_{-r}$  and  $[X_r, X_{-r}]$  together form a copy of  $\mathfrak{sl}(2, \mathbb{C})$  and the structure constants  $N_{\beta, \gamma}$  satisfy the relation  $N_{\beta, \gamma} = -N_{-\beta, -\gamma}$ . The Lie algebra of  $K$  is the fixed point set of the semilinear involution of  $\mathfrak{g}$  that extends the map that sends every  $X_r$  of the above type to  $X_{-r}$  (see [11, p. 102, Lemma 44] and [11, p. 103, Lemma 45]). We fix a choice of roots  $X_r$  as above.

Let  $\omega$  be a closed  $K$ -invariant real form on  $G/P$  of Hodge type  $(1, 1)$ , where  $P$  is a parabolic subgroup. The projection of  $G$  to  $G/P$  will be denoted by  $\pi$ . We have  $H^2(G, \mathbb{R}) = 0$  and  $H^1(G, \mathcal{O}_G) = 0$ , where  $\mathcal{O}_G$  denotes the structure sheaf of holomorphic functions on  $G$ . Consequently, the form

$$\hat{\omega} := \pi^* \omega$$

on  $G$  is of the form  $dd^c \varphi$  for some smooth real valued function  $\varphi$  on  $G$ . In other words,  $\varphi$  is a *quasi-potential* for  $\omega$ . Since  $\omega$  is  $K$ -invariant, the function  $\varphi$  can also be chosen to be  $K$ -invariant. Since  $dd^c \varphi = \hat{\omega}$  and  $R_p^*(\hat{\omega}) = \hat{\omega}$ , where  $p \in P$  and  $R_p$  is the right translation of  $G$ , it follows immediately that

$$\varphi(gp) = \varphi(g) + c(p) \quad (2.2)$$

where  $g \in G$ ,  $p \in P$  and  $c$  is a real valued additive character on  $P$ .

**Proposition 2.1.** *Let  $\psi : G \rightarrow \mathbb{R}$  be a  $K$ -invariant function and  $c : P \rightarrow \mathbb{R}$  a real valued additive character on  $P$  such that  $\psi(gp) = \psi(g) + c(p)$  for all  $g \in G$ . Then there is a form  $\omega$  on  $G/P$  such that the pull back form  $\pi^* \omega$  on  $G$  coincides with  $dd^c \psi$ .*

**Proof.** Let  $U \subset G/P$  be an open subset such that there is a section  $s_U : U \rightarrow G$  over  $U$  of the projection  $\pi$ . Note that for a form  $\omega$  on  $G/P$ , the identity  $\omega = s_U^*(\pi^* \omega)$  is valid. Set

$$\hat{\omega} := dd^c \psi$$

and consider  $s_U^* \hat{\omega}$ . Let  $s_V : V \rightarrow G$  be a section of  $\pi$  over another open subset  $V$  of  $G/P$ . Since  $G/P$  can be covered by open subsets that have a section of  $\pi$  over it, to prove the proposition it suffices to show that the two differential forms  $s_U^* \hat{\omega}$  and  $s_V^* \hat{\omega}$  coincide over  $U \cap V$ .

Take any point  $\xi \in U \cap V$ . So we have  $\xi = s_U(\xi)P = s_V(\xi)P$ . Hence  $s_U(\xi) = s_V(\xi)f(\xi)$ , where  $f : U \cap V \rightarrow P$  is a smooth map. Now, the condition in the proposition gives

$$\psi(s_U(\xi)) = \psi(s_V(\xi)f(\xi)) = \psi(s_V(\xi)) + c(f(\xi)),$$

and hence  $s_U^* \psi = s_V^* \psi + f^*c$ . So to prove that  $s_U^* \hat{\omega} = s_V^* \hat{\omega}$  on  $U \cap V$  it is enough to show that  $dd^c c = 0$ .

Let  $P' := [P, P]$  be the commutator of  $P$  and  $q : P \rightarrow P/P'$  the natural projection. There is an additive character  $\chi$  of  $P/P'$  with  $\chi \circ q = c$ . The group  $P/P'$  is the image by a homomorphism of the additive group  $\mathbb{C}^N$ , for some  $N$ . Since  $dd^c \chi' = 0$  for any character  $\chi'$  of  $\mathbb{C}^N$ , we have  $dd^c c = q^*(dd^c \chi) = 0$ .

Now consider the two-form on  $G/P$  obtained by patching together locally defined forms of the form  $s_U^* \hat{\omega}$ . This form is  $K$ -invariant since  $\psi$  is  $K$ -invariant. This completes the proof of the proposition.  $\square$

By adding a constant function, one can assume that the quasi-potential  $\varphi$  for  $\omega$  vanishes on the commutator  $P'$  of  $P$  at well as on  $K$ . Since  $G = KP$ , the function  $\varphi$  is completely determined by its restriction to  $P$ . The Levi-complement  $L_p$  of  $P$ , which contains  $T$ , is given by a subset  $S_p$  of the simple roots  $S$ . More precisely,

$$P = P' T_1$$

where  $T_1 \subset T$  is as follows:

$$T_1 = \left\{ \prod \check{\alpha}(z_\alpha) \mid \alpha \in S \setminus S_p \text{ and } z_\alpha \in \mathbb{C} \right\} \tag{2.3}$$

where  $\check{\alpha}$  is the one-parameter multiplicative subgroup defined by the homomorphism  $\phi_\alpha$  of  $SL(2, \mathbb{C})$  to  $G$  constructed earlier.

The additive characters of the one-parameter subgroup  $\check{\alpha}$  which are invariant under  $S^1$  are clearly of the form  $\check{\alpha}(z) = c_\alpha \log |z|$ . Consequently, the function  $\varphi$  is determined by the restriction of the character  $c$  in (2.2) to  $T_1$ . As we already noted, the restriction of  $c$  to  $T_1$  is of the form

$$c \left( \prod_{\alpha \in S \setminus S_p} \check{\alpha}(z_\alpha) \right) = \sum_{\alpha \in S \setminus S_p} c_\alpha \log |z_\alpha|$$

where  $c_\alpha$  are constants. We will show that such a character of  $T_1$  can be extended to a quasi-potential on  $G$ . This is a consequence of the following proposition.

**Proposition 2.2.** *Let  $\rho$  be an irreducible representation of  $G$  with highest weight  $\lambda$ . Fix a Hermitian inner product on the representation space which is  $K$ -invariant and let  $v$  be a highest weight vector of norm 1. The function  $\varphi$  on  $G$  defined by*

$$\varphi(g) := \log |\rho(g)v|$$

*is  $K$ -invariant and satisfies the identity in (2.2).*

**Proof.** Since the Hermitian inner product is  $K$ -invariant,  $\varphi$  is evidently  $K$ -invariant.

The stabilizer of the line  $\mathbb{C}v$  is a parabolic subgroup which will be denoted by  $P_v$ . For any  $p \in P_v$  we have  $pv = \chi(p)v$ , where  $\chi$  is a character of  $P_v$ . Now,  $G = KP_v$ , so for any  $k \in K$  and  $p_1, p_2 \in P_v$  we have

$$\log |\rho(kp_1p_2v)| = \log |\chi(p_1p_2)v| = \log |\chi(p_1)| + \log |\chi(p_2)|.$$

We use that  $\log |\chi(p)v| = \log |\chi(p)|$ , which in turn follows from the given condition that  $v$  is of norm 1. This completes the proof of the proposition.  $\square$

Returning to the original setting, for  $\alpha \in S$ , let  $\rho_\alpha$  be the irreducible representation of  $G$  with highest weight  $\omega_\alpha$  defined by

$$\omega_\alpha(\check{\beta}(z)) := z^{\delta(\alpha, \beta)} \tag{2.4}$$

for all  $\beta \in S$ , where  $\delta(\alpha, \beta) = 0$  if  $\alpha \neq \beta$  and  $\delta(\alpha, \alpha) = 1$ .

Set  $\rho$  in Proposition 2.2 to be  $\rho_\alpha$ . Let

$$\varphi_\alpha : G \rightarrow \mathbb{R} \tag{2.5}$$

be the function defined by  $\varphi_\alpha(g) = \log |\rho_\alpha(g)v_\alpha|$ , where  $v_\alpha$  is the highest weight vector of norm 1. In other words, if we set  $v$  in Proposition 2.2 to be a highest weight vector  $v_\alpha$  of norm 1, then the function  $\varphi$  in Proposition 2.2 coincides with  $\varphi_\alpha$  in (2.5). Let  $P_\alpha$  denote the parabolic subgroup of  $G$  that preserves the line defined by  $v_\alpha$ . The function  $\varphi_\alpha$  vanishes on the commutator of  $P_\alpha$  and therefore on the commutator of the Levi-complement of  $P_\alpha$ . Moreover,

$$\varphi_\alpha(\check{\beta}(z)) = \delta(\alpha, \beta) \log |z|.$$

Recall that the Levi-complement  $L_P$  of a parabolic subgroup  $P$  which contains  $T$  is given by a subset  $S_P$  of the simple roots  $S$ , and  $P = P'T_1$ , where  $T_1$  is defined in (2.3). Therefore, the function  $\sum_{\alpha \in S \setminus S_P} c_\alpha \varphi_\alpha$  restricts to the desired additive character on  $T_1$ . Moreover, the function vanishes on  $(L_P)'R_u(P)$ , where  $R_u(P)$  is the unipotent radical of  $P$  and  $(L_P)'$  is the commutator of  $L_P$ .

In summary, in the section we have shown that if  $\omega$  is a closed real  $K$ -invariant  $(1, 1)$  form on  $G/P$ , then its pull back to  $G$  has a  $K$ -invariant potential whose restriction to  $P$  is an additive character of  $P$ . This restriction is invariant under the action of the maximal compact subgroup  $T \cap K$  of  $T$  and it vanishes on the commutator of  $P$ . Therefore, it is

completely described by its values on  $T$  and the potential function is a linear combination of functions of the form  $\log |\rho(g)v|$ , where  $\rho$  is an irreducible representation of  $G$  and  $v$  is a highest weight vector of norm 1 in this representation space. Moreover, any linear combination of such functions qualifies as a quasi-potential for a  $K$ -invariant closed real  $(1, 1)$  form on  $G/P$ .

### 3. Nondegenerate forms

The aim in this section is to determine which linear combinations of the quasi-potentials  $\varphi_\alpha$  (constructed in (2.5)) define a Kähler structure on  $G/P$ .

Let  $\omega$  be a real  $(1, 1)$  form on a complex manifold  $M$ . It defines a Hermitian form  $h_\omega$  on the holomorphic tangent bundle  $T^{(1,0)}$  which is given by  $h_\omega(X, Y) = -\sqrt{-1}\omega(X, \bar{Y})$ . If  $\omega$  is also closed with local potential  $\varphi$ , then  $h_\omega(X, Y) = \partial\bar{\partial}\varphi(X, \bar{Y})$ . The form  $\omega$  is *nondegenerate* (respectively, *positive*) if the corresponding Hermitian form is nondegenerate (respectively, positive). Using the notation of Section 2, a criterion for a quasi-potential  $\varphi$  to define a nondegenerate (or positive) form on  $G/P$  is given by the following result.

**Theorem 3.1.** *The quasi-potential*

$$\varphi := \sum_{\alpha \in S \setminus S_P} c_\alpha \varphi_\alpha,$$

where  $\varphi_\alpha$  is defined in (2.5) and  $c_\alpha \in \mathbb{R}$ , defines a positive form  $\omega_\varphi = \omega$  on  $G/P$  if and only if all the constants  $c_\alpha$  are positive.

It defines a nondegenerate form if and only if for all roots  $\beta \in R_u(P)$ , we have  $\sum_{\alpha \in S \setminus S_P} c_\alpha \omega_\alpha(\check{\beta}) \neq 0$ .

**Proof.** The group  $K$  operates transitively on  $G/P$ . Since  $\omega_\varphi$  is  $K$ -invariant, it suffices to check the assertions in the theorem only at the point  $\xi_0 = eP$ , where  $e \in G$  is the identity element.

Let  $R_u(\bar{P})$  be the unipotent radical of the opposite parabolic group  $\bar{P}$ . Now the map from  $R_u(\bar{P})$  to  $G/P$  defined by  $r \mapsto r\xi_0$  is an isomorphism onto an open neighborhood, say  $U$ , of  $\xi_0$  in  $G/P$ . So its inverse gives a local section, say  $s$ , over  $U$  of the projection  $\pi$  of  $G$ , and  $\omega_U = \sqrt{-1}\partial\bar{\partial}(s^*\varphi)$ .

For a root  $\beta \in R_u(P)$ , let  $U_{-\beta}$  be the corresponding root group of the opposite parabolic group  $\bar{P}$ . By applying the Gram–Schmidt process to  $\text{SL}(2, \mathbb{C})$  and transposing the calculation to  $P$  (cf. [1, Proposition 2.1]), we see that the restriction of  $\omega$  to the line  $U_{-\beta}\xi_0$  coincides with

$$\sqrt{-1}\partial\bar{\partial} \sum_{\alpha \in S \setminus S_P} c_\alpha \log |\rho_\alpha(U_{-\beta}(z)v_\alpha)| = \sqrt{-1}\partial\bar{\partial} \sum_{\alpha \in S \setminus S_P} \frac{c_\alpha}{2} \omega_\alpha(\check{\beta}) \log(1 + |z|^2),$$

where  $\omega_\alpha$  is defined in (2.4). Consider the tangent vectors  $\frac{\partial}{\partial z}|_{z=0}U_{-\beta}(z)\xi_0 = X_\beta^*$  with  $\beta \in R_u(P)$ . These vectors span the tangent space  $T_{\xi_0}^{1,0}G/P$ . These are eigen vectors for the action of  $T \cap K$  with different weights  $-\beta$ . Therefore, the vectors  $X_\beta^*$ , where  $\beta \in R_u(P)$ , are orthogonal relative to any  $T \cap K$  invariant Hermitian form.

Moreover, we have

$$h_\omega(X_\beta^*, X_\beta^*) = \sum_{\alpha \in S \setminus S_p} \frac{c_\alpha}{2} \omega_\alpha(\check{\beta}).$$

For  $\beta \in S \setminus S_p$ , we have  $h_\omega(X_\beta^*, X_\beta^*) = c_\beta/2$ . Also, for  $\beta \in R_u(P)$ , the integer  $\omega_\alpha(\check{\beta})$  is positive for some  $\alpha \in S \setminus S_p$ . Consequently, the form  $\omega$  is positive if and only if all the constants  $c_\alpha$ , where  $\alpha \in S \setminus S_p$ , are positive. It is nondegenerate if and only if for every root  $\beta \in R_u(P)$ , we have  $\sum_{\alpha \in S \setminus S_p} c_\alpha/2\omega_\alpha(\check{\beta}) \neq 0$ . This completes the proof of the theorem.  $\square$

#### 4. Ricci form of a volume form

In this section, after recalling the basic definition of a Kähler–Einstein form, we recall from [3] a description of the first Chern class of line bundles over  $G/P$ .

Let  $V$  be a volume form on a complex manifold  $M$  of dimension  $n$ . If  $\eta$  is a locally defined nonvanishing holomorphic  $n$  form on  $M$ , then  $V = \varphi\eta\bar{\eta}$  for some function  $\varphi$ . The *Ricci form* of  $V$  is defined to be

$$\text{Ric}(V) := -\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log|\varphi|. \quad (4.1)$$

On the other hand, if  $L$  is a holomorphic line bundle over  $M$  equipped with a Hermitian structure  $N$  and  $s$  a locally defined nonvanishing holomorphic section of  $L$ , then the first Chern class of  $L$  is defined by

$$c_1(L) := -\frac{\sqrt{-1}}{\pi} \partial\bar{\partial} \log|N(s)|.$$

Let  $M = \bigcup_i U_i$  be a covering of  $M$  by open subsets on which there is a nonvanishing holomorphic section, say  $s_{U_i}$ , of  $L$ . On  $U_i \cap U_j$ , set  $g_{ij} = s_{U_i}/s_{U_j}$ . A Hermitian structure on  $L$  is given by a family of real positive functions  $\lambda_i$  defined on each  $U_i$  with  $\lambda_i/\lambda_j = |g_{ij}|$ .

Therefore, a Ricci form as in (4.1) is the first Chern form of the anti-canonical line bundle of  $M$  equipped with the Hermitian structure induced by the volume form  $V$ . In particular the Ricci form represents the first Chern class  $c_1(M)$  (cf. [6]). Also, if  $f: M_1 \rightarrow M$  is a holomorphic map which is locally an isomorphism, then  $f^*V$  is a volume form on  $M_1$ , where  $V$  is a volume form on  $M$ . Since  $\text{Ric}(f^*V) = f^*\text{Ric}(V)$ , a holomorphic map of  $M$  to itself that preserves  $V$  must also preserve the Ricci form.

A Kähler form  $\omega$  on  $M$  is called *Kähler–Einstein* if the Ricci form of  $\omega^n$  is a constant scalar multiple of  $\omega$ . The constant is known as *Einstein constant*.

Now let  $M$  be the flag manifold  $G/P$ . For any  $\alpha \in S \setminus S_P$ , consider the function  $f_\alpha$  on  $G$  defined by

$$g \mapsto \frac{1}{\pi} \log |\rho_\alpha(g)v_\alpha|,$$

where  $\rho_\alpha$  is the irreducible representation of  $G$  with highest weight  $\omega_\alpha$  and  $v_\alpha$  a highest weight vector of norm 1 therein. Proposition 2.2 says that the form  $dd^c f_\alpha$  descends to  $G/P$ . In other words, there is a (unique)  $(1, 1)$ -form  $\omega_\alpha$  on  $G/P$  such that the pull back of  $\omega_\alpha$  to  $G$  coincides with  $dd^c f_\alpha$ . From Theorem 3.1 it follows that  $\omega_\alpha$  is a positive form on  $G/P$ .

Every holomorphic character  $\chi$  of  $P$  defines a holomorphic line bundle  $L_\chi$  on  $G/P$ . The first Chern class of  $L_\chi$  has the expression

$$c_1(L_\chi) := - \sum_{\alpha \in S \setminus S_P} \langle \chi, \check{\alpha} \rangle [\omega_\alpha],$$

where  $\omega_\alpha$ , as above, is the form on  $G/P$  defined by  $dd^c f_\alpha$  (see [3]). Furthermore, the cohomology class  $-\langle \chi, \check{\alpha} \rangle [\omega_\alpha]$  is dual to the projective line in  $G/P$  defined by  $L_\alpha \xi_0$  in  $G/P$ , where  $L_\alpha$ , as in Section 2, is the image of  $SL(2, \mathbb{C})$  by the map  $\phi_\alpha$  defined in (2.1) [3]. Moreover, the first Chern class of the anti-canonical line bundle is

$$c_1(G/P) = \sum_{\alpha \in S \setminus S_P} \langle \rho, \check{\alpha} \rangle [\omega_\alpha] \tag{4.2}$$

where  $\rho$  is the sum of all positive roots not supported by  $S_P$ ; the integers  $\langle \rho, \check{\alpha} \rangle$  are positive. For a complex manifold  $M$ , by  $c_1(M)$  we mean  $c_1(TM)$ .

### 5. Kähler–Einstein metrics on $G/P$

In this section we consider a natural Kähler form on  $G/P$ , show that it defines a Kähler–Einstein structure on  $G/P$ , and prove that the restriction of it to any fiber of the projection  $G/P \rightarrow G/Q$ , where  $Q \supset P$  is a parabolic subgroup of  $G$ , is also a Kähler–Einstein metric.

Using the notation of Section 4, let  $\omega$  be the form  $\sum_{\alpha \in S \setminus S_P} \langle \rho, \check{\alpha} \rangle \omega_\alpha$  on  $G/P$ . Since  $\langle \rho, \check{\alpha} \rangle$  are positive integers, we see from Theorem 3.1 that  $\omega$  is a positive form. Since  $\omega$  is  $K$ -invariant, the corresponding volume form and therefore its Ricci form are also  $K$ -invariant. Note that from (4.2) it follows that the two  $(1, 1)$ -forms  $\omega$  and  $\text{Ric}(\omega)$  are cohomologous.

Therefore,  $\omega$  and  $\text{Ric}(\omega)$  are  $K$ -invariant cohomologous forms on  $G/P = K\xi_0$ . Firstly, since  $\omega - \text{Ric}(\omega)$  is a real  $d$ -exact form of type  $(1, 1)$ , the  $dd^c$ -lemma (see [4, p. 85, Corollary 2.110]) says that  $\omega - \text{Ric}(\omega) = dd^c f$ . Secondly, since any harmonic function on  $G/P$  is a constant function and  $\omega - \text{Ric}(\omega)$  is  $K$ -invariant, it follows that the function  $f$  is  $K$ -invariant. Finally, since  $K$  acts transitively on  $G/P$ , we conclude that  $\omega = \text{Ric}(\omega)$ . In other words,  $\omega$  is a Kähler–Einstein metric with Einstein constant 1.



We will call  $\omega$  the *canonical metric* on  $G/P$ .

**Theorem 5.1.** *Let  $Q$  be a parabolic subgroup of  $G$  containing  $P$ . Then the restriction of the canonical metric  $\omega$  to any fiber of the projection  $G/P \rightarrow G/Q$  is a Kähler–Einstein metric.*

**Proof.** Let  $S_P$  and  $S_Q$  be the roots of the Levi-complements  $L_P$  and  $L_Q$ , of  $P$  and  $Q$  respectively, which contain  $T$ . The fibers of the projection  $G/P \rightarrow G/Q$  are  $K$ -translates of the fiber  $Q/P$ , which itself is a flag space of the reductive group  $L_Q$ . Therefore,  $c_1(Q/P)$  is given by the restriction

$$c_1(Q/P) = c_1(L_Q/(L_Q \cap P)) = \sum_{\alpha \in R} \langle \sigma, \check{\alpha} \rangle [\omega_\alpha]|_{Q/P} \quad (5.1)$$

where  $R = S_Q \setminus S_P$  and  $\sigma$  is the sum of all positive roots with support outside  $S_P$  but which are supported by  $S_Q$ .

Consider the form

$$\omega = \sum_{\alpha \in S \setminus S_P} \langle \rho, \check{\alpha} \rangle \omega_\alpha$$

defining the canonical metric on  $G/P$ , which is a Kähler–Einstein metric with Einstein constant 1. So the restriction of  $\omega$  to  $Q/P$  has the expression

$$\omega|_{Q/P} = \sum_{\alpha \in R} \langle \rho, \check{\alpha} \rangle \omega_\alpha|_{Q/P} + \sum_{\alpha \in \tilde{R}} \langle \rho, \check{\alpha} \rangle \omega_\alpha|_{Q/P} \quad (5.2)$$

where  $\tilde{R}$  is the complement of  $S_Q$  in the set  $S$  of all simple roots.

Now, the second sum in the right-hand side of (5.2) vanishes identically. Indeed,  $Q/P$  is a homogeneous space for the simple group  $[Q, Q]$ . Now, from the summary at the end of Section 2 it follows that the forms  $\omega_\alpha$ , where  $\alpha \in \tilde{R}$ , vanish on  $Q/P$  because the corresponding quasi-potential is zero on the commutator  $[L_Q, L_Q]$ . Since the second sum in the right-hand side of (5.2) is zero, we have

$$\omega|_{Q/P} = \sum_{\alpha \in R} \langle \rho, \check{\alpha} \rangle \omega_\alpha|_{Q/P}.$$

Now,  $\rho = \sigma + \tau$ , where  $\tau$  is the sum of all positive roots with support outside  $S_P$ , but which are supported by  $S_Q$ , and  $\sigma$  is the sum of all positive roots whose support lies outside  $S_Q$ . Consequently, if  $\alpha \in S_Q$ , the reflection along  $\alpha$  fixes  $\tau$ . Therefore, we have  $\langle \rho, \check{\alpha} \rangle = \langle \sigma, \check{\alpha} \rangle$ . Hence

$$\omega|_{Q/P} = \sum_{\alpha \in R} \langle \sigma, \check{\alpha} \rangle \omega_\alpha|_{Q/P}. \quad (5.3)$$

Combining (5.1) and (5.3) we see that  $\omega|_{Q/P}$  represents  $c_1(Q/P)$  and as it is invariant under  $K \cap Q$ , we use the  $dd^c$ -lemma as before to conclude that  $\omega|_{Q/P}$  is a Kähler–Einstein metric with Einstein constant 1. This completes the proof of the theorem.  $\square$

### 6. Application to volume computation

Our aim in this final section is to compute the volume of  $G/P$  with respect to the canonical metric. We start with some special cases.

Let  $\text{Gr}$  denote the Grassmann variety parametrizing the space of all  $d + 1$  dimensional linear subspaces of  $\mathbb{C}^{n+1}$ , with  $0 < d + 1 \leq n$ . Set  $N = (d + 1)(n - d)$  which is the dimension of  $\text{Gr}$ .

Let  $S$  denote the tautological vector bundle over  $\text{Gr}$  whose fiber over the point representing a subspace  $V$  of  $\mathbb{C}^{n+1}$  is  $V$  itself. Set  $\tau = -c_1(S)$  which is a Kähler class on  $\text{Gr}$ . Note that  $H^2(\text{Gr}, \mathbb{Z}) = \mathbb{Z}$  and  $\tau$  is the positive generator.

Recall the Euler sequence

$$0 \rightarrow \text{End}(S) \rightarrow \text{Hom}(S, \underline{\mathbb{C}}^{n+1}) \rightarrow T \text{Gr} \rightarrow 0$$

over  $\text{Gr}$ , where  $\underline{\mathbb{C}}^{n+1}$  is the trivial vector bundle with fiber  $\mathbb{C}^{n+1}$ . The Euler sequence implies

$$c_1(\text{Gr}) = (n + 1)\tau. \tag{6.1}$$

The evaluation of the cohomology class  $\tau^N$  on the top (oriented) homology class  $\text{Gr}$  is

$$A := \frac{1!2! \cdots N!}{(n - d)!(n - d + 1)! \cdots n!} \tag{6.2}$$

[7, p. 274, Eq. iii]. This combining with (6.1) give

$$c_1(T \text{Gr})^N \cap [\text{Gr}] = A(n + 1)^N \tag{6.3}$$

where  $\cap$  denotes the *cap* product.

The canonical metric (defined in Section 5) is Kähler–Einstein with Einstein constant 1. In particular, the second cohomology class represented by the canonical metric  $\omega$  on  $\text{Gr}$  coincides with  $c_1(T \text{Gr})$ . The total volume of  $\text{Gr}$  with respect to  $\omega$  is the integral of  $\omega^N$  over  $\text{Gr}$ . Consequently, (6.3) implies that the total volume of  $\text{Gr}$  with respect to  $\omega$  is  $A(n + 1)^N$ , where  $A$  is the constant defined in (6.2).

We will now compute the volume of  $\text{GL}(n, \mathbb{C})/B$ , where  $B$  is a Borel subgroup. The dimension of  $\text{GL}(n, \mathbb{C})/B$  is  $\frac{(n-1)n}{2}$ .

From [8, p. 282, Theorem 5] we know that the first Chern class of the holomorphic tangent bundle is  $c_1(T(\mathrm{GL}(n, \mathbb{C})/B)) = 2 \sum_{i=0}^{n-2} A_i$  (using the notation of [8]). On the other hand, [8, p. 260, Theorem 1] implies that

$$\left( \sum_{i=0}^{n-2} A_i \right)^{\frac{(n-1)n}{2}} = \left( \frac{(n-1)n}{2} \right)!$$

(see also [8, p. 277, (11.2)] and the lines following it). Combining these two, it follows that

$$c_1(T(\mathrm{GL}(n, \mathbb{C})/B))^{\frac{(n-1)n}{2}} = 2^{\frac{(n-1)n}{2}} \left( \frac{(n-1)n}{2} \right)!.$$

Since the Einstein constant of the canonical metric is 1, the total volume of  $\mathrm{GL}(n, \mathbb{C})/B$  with respect to the canonical metric is  $2^{\frac{(n-1)n}{2}} \left( \frac{(n-1)n}{2} \right)!$ .

The volume of general  $G/P$  can be computed using [5, p. 340, Theorem 24.10]. The weight  $d$  (in the notation of [5, Theorem 24.10]) for the holomorphic tangent bundle  $TG/P$  is given in (4.2). If we set  $d$  in [5, Theorem 24.10] to be the weight for the holomorphic tangent bundle  $TG/P$ , then [5, Theorem 24.10] gives the degree  $c_1(G/P)^{\dim G/P} \in \mathbb{Z}$  of the tangent bundle.

We note a general fact that if  $V_M$  be a volume form on a projective manifold  $M$  of complex dimension  $m$  such that the element in  $H^{2m}(M, \mathbb{R})$  represented by  $V_M$  coincides with  $c_1(M)^m$  and, furthermore, if the line bundle  $\bigwedge^m TM$  is very ample, then the degree of the embedding of  $M$  in  $\mathbb{P}H^0(M, \bigwedge^m TM)^*$  coincides with the volume of  $M$  with respect to  $V_M$ . It is easy to see that all the conditions on the pair  $(M, V_M)$  are satisfied by the canonical metric on  $G/P$ . Indeed, this is an immediate consequence of the fact that the canonical metric is Kähler–Einstein with Einstein constant 1.

Therefore, from [5, Theorem 24.10] the total volume of  $G/P$  with respect to the canonical metric is

$$n! \prod_{\alpha \in S} (d, \alpha)(a, \alpha)$$

where  $n = \dim G/P$  and  $a$  is the sum of all positive roots.

By Theorem 3.1, if  $V$  is the volume form on  $G/P$  corresponding to the form

$$\sum_{\alpha \in S \setminus S_P} c_\alpha \omega_\alpha$$

and  $V_0$  is the canonical volume form on  $G/P$ , then  $V/V_0$  is given by

$$\frac{V}{V_0} = \frac{\prod_{\beta \in R_u(P)} \sum_{\alpha \in S \setminus S_P} c_\alpha \omega_\alpha(\check{\beta})}{\prod_{\beta \in R_u(P)} \sum_{\alpha \in S \setminus S_P} \langle \rho, \check{\alpha} \rangle \omega_\alpha(\check{\beta})}.$$

Therefore, from the above computation of the total volume for  $V_0$  we get the total volume of  $V$  as well.

We put down the result of the above computations in the form of the following theorem.

**Theorem 6.1.** *The volume of  $G/P$  with respect to the canonical metric is*

$$V_0 := n! \prod_{\alpha \in S} (d, \alpha)(a, \alpha)$$

where  $n = \dim G/P$ ,  $d = \sum_{\alpha \in S \setminus S_P} \langle \rho, \check{\alpha} \rangle [\omega_\alpha]$ , and  $a$  is the sum of all positive roots. More generally, the volume of  $G/P$  with respect to the form  $\sum_{\alpha \in S \setminus S_P} c_\alpha \omega_\alpha$  is

$$\frac{V_0 \prod_{\beta \in R_u(P)} \sum_{\alpha \in S \setminus S_P} c_\alpha \omega_\alpha(\check{\beta})}{\prod_{\beta \in R_u(P)} \sum_{\alpha \in S \setminus S_P} \langle \rho, \check{\alpha} \rangle \omega_\alpha(\check{\beta})}.$$

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