Group classification, optimal system and optimal reductions of a class of Klein Gordon equations

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Abstract

Complete symmetry analysis is presented for non-linear Klein Gordon equations $u_{tt} = u_{xx} + f(u)$. A group classification is carried out by finding $f(u)$ that give larger symmetry algebra. One-dimensional optimal system is determined for symmetry algebras obtained through group classification. The subalgebras in one-dimensional optimal system and their conjugacy classes in the corresponding normalizers are employed to obtain, up to conjugacy, all reductions of equation by 2-dimensional subalgebras. This is a new idea which improves the computational complexity involved in finding all possible reductions of a PDE of the form $F(x, t, u, u_x, u_t, u_{xx}, u_{tt}, u_{xt}) = 0$ to a first order ODE. Some exact solutions are also found.

Key words: Nonlinear wave equation, Lie symmetries, group classification, optimal system, invariant solutions.

1 Introduction

This paper gives a complete symmetry analysis of a class of non-linear wave equations. We follow ideas of Ovsiannikov [15] , Ibragimov [6, 7] and Clarkson-Mansfield [12] to carry this through.

The equations considered are the non-linear Klein Gordon equations in one space dimension, namely the non-linear wave equations of the form

$$u_{tt} = u_{xx} + f(u). \qquad (f_{uu} \neq 0) \tag{1.1}$$

The analysis consists of first finding the Lie symmetries of equation with arbitrary $f(u)$ and then determining all possible forms of $f(u)$ for which larger symmetry groups exist.

This is followed by the determination of optimal systems of subalgebras and reductions and invariant solutions corresponding to these optimal systems.

The first group classification problem was carried out by Ovsiannikov [15] who classified all forms of the non-linear heat equation $u_t = (f(u)u_x)_x$. Studies related to group properties of non-linear wave equations began with the well-known paper of Ames [1] in 1981.

The group classification problem for the equation $u_{tt} = u_{xx} + u_{yy} + u_{zz} + f(u)$ in three space dimensions has been carried out by Rudra in [16] . However, we have followed ideas of Clarkson-Mansfield [12] to carry this through because of the link with Groebner bases and their potential wider applicability.

The calculations of minimal symmetry algebra and the forms of $f(u)$ which provide larger symmetry algebra are based on necessary conditions on $f(u)$ obtained through a *triangulation* of determining equations of Lie symmetries of Equation (1.1). An efficient method to obtain *triangulation* is the well-known method of Mansfield [13, Section 2.9] of generating differential Gröbner bases of determining equations using Kolchin-Ritt algorithm.

Here we obtain a *triangulation* of the determining equations using a variant of Kolchin-Ritt algorithm, the *direct search algorithm* by Clarkson-Mansfield [12], that allows an efficient triangulation of determining equations: the reader is referred to [12] for a detailed discussion to carry out calculations based on these algorithms.

For the determination of optimal subalgebras and corresponding reductions of PDE (1.1) to a first order ODE, we have used a variant of schemes of Ibragimov [7] and Ovsiannikov [15], and have introduced a new idea. Generally, for all possible reductions of PDE (1.1) to a first order ODE, one needs conjugacy classes of 2-dimensional subalgebras. Here to improve the computational methodology, we have first determined conjugacy classes of 1-dimensional subalgebras and reduced the PDE to a second order ODE using these classes. And then (where needed) have carried out the second reductions to reduce PDE (1.1) to a first order ODE by determining and utilizing the conjugacy classes in the normalizers of 1-dimensional subalgebras in 1-dimensional optimal system. The optimal system of 1-dimensional subalgebras is obtained in Section 3, and the implementation of the new idea for obtaining desired reductions and solutions is carried out in Section 4.

The literature on classical Lie symmetry theory, its applications and its extensions is vast; the reader is referred to [2, 3, 4, 5, 8, 9, 10, 11, 14, 15].

2 Symmetry group classification

In this section we present complete classification of Lie symmetries of Equation (1.1). To obtain the Lie symmetries of Equation (1.1) we consider the one parameter Lie group of infinitesimal transformations in (x, t, u) given by

$$x^* = x + \epsilon\xi(x, t, u) + O(\epsilon^2)$$

$$t^* = t + \epsilon\tau(x, t, u) + O(\epsilon^2)$$

$$u^* = u + \epsilon\phi(x, t, u) + O(\epsilon^2)$$

where ϵ is the group parameter, hence the corresponding generator of the Lie algebra is of the form

$$X = \xi(x, t, u)\frac{\partial}{\partial x} + \tau(x, t, u)\frac{\partial}{\partial y} + \phi(x, t, u)\frac{\partial}{\partial u}.$$

If $X^{[2]}$ denotes the second prolongation of X then using the invariance condition

$$X^{[2]}\left(u_{tt} - u_{xx} - f(u)\right)|_{u_{tt}=u_{xx}+f(u)} = 0$$

yields the following system of 8 determining equations.

$$e_1: \quad \xi_u = 0$$

$$e_2: \quad \tau_u = 0$$

$$e_3: \quad -\xi_x + \tau_t = 0$$

$$e_4: \quad \xi_t - \tau_x = 0$$

$$e_5: \quad \phi_{uu} = 0$$

$$e_6: \quad -\tau_{tt} + \tau_{xx} + 2\phi_{tu} = 0$$

$$e_7: \quad -\xi_{tt} + \xi_{xx} - 2\phi_{xu} = 0$$

$$e_8: \quad -f_u\phi - 2f\tau_t + f\phi_u + \phi_{tt} - \phi_{xx} = 0$$

The main tool to obtain triangulation of determining equations are the operations of finding *diffSpolynomial* of two differential polynomials and *pseudo-reduction* of a differential polynomial by a set of differential polynomials; we refer to [13, 12] for details and background.

Assuming the lexicographic ordering of the system as $\phi > \tau > \xi > f$ and $x > t > u$. The *diffSpolynomial* of e_3 and e_4 gives

$$e_9: \quad \xi_{tt} - \xi_{xx} = 0$$

which directly leads to

$$e_{10}: \quad \tau_{tt} - \tau_{xx} = 0.$$

Differentiating e_8 twice with respect to u and pseudo-reducing, by $\{e_{11}, e_{12}\}$, the derivative of e_{12} with respect to u yields

$$\begin{aligned} e_{11}: &\quad f_{uu}\phi + 2f_u\tau_t = 0 \\ e_{12}: &\quad f_{uuu}\phi + f_{uu}\phi_u + 2f_{uu}\tau_t = 0 \\ e_{13}: &\quad \left\{f_u f_{uu} f_{uuuu} - 2f_u f_{uuu}^2 + f_{uu}^2 f_{uuu}\right\}\tau_t = 0. \end{aligned}$$

The derivative of e_{11} with respect to t after reduction leads to

$$e_{14}: \quad \left\{f_{uu}^2 - f_u f_{uuu}\right\}\tau_{tt} = 0.$$

Next we look at possibilities for $f(u)$. If $\tau_t = 0$ then by $e_{11},\ e_{10},\ e_3$ and e_4 we obtain

$$\begin{aligned} &\phi = 0 \\ &\tau = k_2 + k_3 x \\ &\xi = k_1 + k_3 t. \end{aligned}$$

without any restriction on $f(u)$. Hence the minimal symmetry algebra is 3-dimensional which exits for any choice of $f(u)$ and is spanned by

$$t\frac{\partial}{\partial x} + x\frac{\partial}{\partial t}, \quad \frac{\partial}{\partial t}, \quad \frac{\partial}{\partial x}.$$

To look for functions $f(u)$ that may give larger symmetry algebra we assume $\tau_t \neq 0$ and solve the differential equations

$$f_{uu}^2 - f_u f_{uuu} = 0 \tag{2.1}$$

and

$$f_u f_{uu} f_{uuuu} - 2f_u f_{uuu}^2 + f_{uu}^2 f_{uuu} = 0. \tag{2.2}$$

The substitution $H = f_u$ reduces Equation (2.1) to

$$\frac{H_u}{H} - \frac{H_{uu}}{H_u} = 0$$

which gives $f(u) = ae^{bu} + c$.
The substitution $H = f_u$ reduces Equation (2.2) to

$$\frac{H_{uuu}}{H_{uu}} - 2\frac{H_{uu}}{H_u} + \frac{H_u}{H} = 0$$

which gives

$$(i) \quad f(u) = au^2 + bu + c$$

$$(ii) \quad f(u) = (au + b)^n + c \quad \text{for } n \neq 0, 1, 2$$

$$(iii) \quad f(u) = \tfrac{\ln(au+b)}{a} + c$$

$$(iv) \quad f(u) = ae^{bu} + c.$$

Hence from e_{14} we see that if $f(u) \neq ae^{bu} + c$, we must have $\tau_{tt} = 0$.

The symmetry algebras for different forms of $f(u)$ are summarized in the following cases.

2.1 $f(u) \neq ae^{bu} + c$

2.1.1 $f(u) = u^2 + bu + c$

It follows from Equations e_{11} and e_8 that $(b^2 - 4c)\tau_t = 0$. Hence a larger symmetry algebra is possible if $b^2 - 4c = 0$ i.e. $f(u)$ is a perfect square. For $f(u) = u^2$ the symmetry algebra is 4-dimensional and is determined by

$$\xi = k_1 t + k_3 + k_4 x$$

$$\tau = k_1 x + k_2 + k_4 t$$

$$\phi = -2uk_4.$$

2.1.2 $f(u) = u^n + c \; n \neq 0, 1, 2$

Equations e_{11} and e_8 imply that $c \neq 0$ leads to minimal algebra. For $f(u) = u^n$ the symmetry algebra is 4-dimensional and is determined by

$$\xi = k_1 t + k_3 + k_4 x$$

$$\tau = k_1 x + k_2 + k_4 t$$

$$\phi = -\tfrac{2u}{n-1} k_4.$$

2.1.3 $f(u) = \ln(u + b) + c$

Equations e_{11} and e_8 imply that $\tau_t = 0$ which leads to minimal algebra.

2.2 $f(u) = ae^{bu} + c$

Here it is not necessary to have $\tau_{tt} = 0$ but using Equations e_{11} and e_8 it follows that unless $c = 0$ there will only be minimal symmetry algebra. For $f(u) = e^{2u}$, Equations $e_{11},\ e_2,\ e_3$ imply that

$$\begin{aligned} \phi &= B(x,t) \\ \tau_t &= -B(x,t) \\ \xi_x &= -B(x,t) \end{aligned} \tag{2.3}$$

where it follows from e_9 or e_{10} that $B(x,t)$ satisfies

$$B_{tt} - B_{xx} = 0. \tag{2.4}$$

The symmetry algebras for simpler forms of $B(x,t)$ are summarized in the table below.

$B(x,t)$	ξ	τ	ϕ
C (constant)	$-k_1 x + k_2 t + k_4$	$-k_1 t + k_2 x + k_3$	k_1
$B(x)$	$-\frac{1}{2}(x^2 + t^2)k_1 - k_2 x + k_3 t + k_5$	$-xtk_1 - k_2 t + k_3 x + k_4$	$k_1 x + k_2$
$B(t)$	$-xtk_1 - k_2 x + k_3 t + k_4$	$-\frac{1}{2}(x^2 + t^2)k_1 - k_2 t + k_3 x + k_5$	$k_1 t + k_2$

In general if $B(x,t)$ is a function such that $B_x \neq 0$ and $B_t \neq 0$ then by Equation (2.4) it must be of the form

$$B(x,t) = f(t + x) - g(t - x). \tag{2.5}$$

Hence, by Equation (2.3),

$$\phi = f(t + x) - g(t - x). \tag{2.6}$$

and

$$\tau_t = -\left[f(t+x) + g(t-x)\right]. \tag{2.7}$$

which gives

$$\tau = -\int \left[f(t+x) + g(t-x)\right] dt + G(x). \tag{2.8}$$

It follows from Equations (2.7), (2.8) and e_{10} that

$$G''(x) = \psi(x) \tag{2.9}$$

where

$$\psi(x) = -\left[f'(t+x) + g'(t-x)\right] + \int \left[f''(t+x) + g''(t-x)\right] dt.$$

Thus τ is determined as

$$\tau = -\int \left[f(t+x) + g(t-x)\right] dt + \int_{a_1}^{x} \left(\int_{a_2}^{u} \psi(s)ds\right) du + k_1 x + k_2. \tag{2.10}$$

From Equation (2.3)

$$\xi = -\int \left[f(t+x) + g(t-x)\right] dt + H(t). \tag{2.11}$$

It follows from e_4 and Equation (2.10) that

$$H'(t) = \rho(t) + k_1 \tag{2.12}$$

where

$$\rho(t) = -\left[f'(t+x) - g'(t-x)\right] + \int \left[f'(t+x) + g'(t-x)\right] dx + \int_{a_1}^{x} \psi(s)ds.$$

This leads to

$$\xi = -\int \left[f(t+x) + g(t-x)\right] dt + \int_{a}^{t} \rho(s)ds + k_1 t + k_3. \tag{2.13}$$

Thus the symmetry algebra is infinite dimensional with the finite dimensional subalgebra consisting of the minimal symmetry algebra.

3 Optimal system of subalgebras

In order to perform symmetry reductions of Equation (1.1) in a systematic manner, we need to obtain a classification of the subalgebras of the symmetry algebra into conjugacy classes under the adjoint action of the symmetry group. We use a variation of the schemes of Ibragimov [7] and Ovsiannikov [15] in the following manner.

The algebras for different cases obtained in Section 2 turn out to be solvable. For reductions, it suffices to determine conjugacy classes of 1-dimensional subalgebras and the conjugacy classes in their normalizers.

The first reduction by subalgebras in optimal system of 1-dimensional subalgebras followed by the reductions by classes of corresponding normalizer will thus give reductions of Equation (1.1) by 2-dimensional subalgebras (up to conjugacy). The optimal system of 1-dimensional subalgebras are obtained in subsequent subsections and the reductions are carried out in detail in Section 4.

3.1 Optimal system of 1-dimensional subalgebras of the symmetry algebra of Equation (1.1) for $f(u) = u^n$ $(n \neq 0, 1)$

This section presents a classification of 1-dimensional subalgebras of symmetry algebras obtained in Section 2.1 for $f(u) = u^n$ $(n \neq 0, 1)$. In each case the symmetry algebra $\mathcal{G}$ is 4-dimensional and is spanned by X_1, X_2, X_3, X_4 where

$$\begin{aligned} X_1 &= t\frac{\partial}{\partial x} + x\frac{\partial}{\partial t}, \\ X_2 &= \frac{\partial}{\partial t}, \\ X_3 &= \frac{\partial}{\partial x}, \\ X_4 &= x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{2u}{n-1}\frac{\partial}{\partial u}. \end{aligned}$$

The commutation relations are

	X_1	X_2	X_3	X_4
X_1	0	$-X_3$	$-X_2$	0
X_2	X_3	0	0	X_2
X_3	X_2	0	0	X_3
X_4	0	$-X_2$	$-X_3$	0

Commutator table for the Lie algebra $\mathcal{G}$.

Set $\mathcal{H} = \langle X_1, X_4 \rangle$ and $\mathcal{G}' = \langle X_2, X_3 \rangle$. Since the commutator $\mathcal{G}'$ is abelian, $\mathcal{G}$ is solvable. It is straight forward to see that

$$e^{t\mathrm{ad}X_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh t & -\sinh t & 0 \\ 0 & -\sinh t & \cosh t & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$e^{t\mathrm{ad}X_4} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{-t} & 0 & 0 \\ 0 & 0 & e^{-t} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$e^{t\,\mathrm{ad}X_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & t \\ t & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and

$$e^{t\,\mathrm{ad}X_3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ t & 1 & 0 & 0 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let $X = \sum_{i=1}^{4} x_i X_i$ be an element of $\mathcal{G}$. Writing $X = (x_1, x_2, x_3, x_4)$, we see that

$$e^{s\,\mathrm{ad}X_2} \cdot e^{t\,\mathrm{ad}X_3}(x_1, x_2, x_3, x_4) = (x_1, tx_1 + x_2 + sx_4, sx_1 + x_3 + tx_4, x_4).$$

The equations

$$tx_1 + x_2 + sx_4 = 0$$

$$tx_4 + x_3 + sx_1 = 0$$

can be solved if $x_1^2 - x_4^2 \neq 0$. So any such element X is conjugate to $(x_1, 0, 0, x_4)$ whenever $x_1^2 - x_4^2 \neq 0$. Thus we get representatives

$$\langle 1, 0, 0, x\rangle \qquad \text{and} \qquad \langle x, 0, 0, 1\rangle \quad (x^2 \neq 1)$$

which are in different conjugacy classes as $\mathcal{G}/\mathcal{G}'$ is abelian.

Assuming $x_1^2 - x_4^2 = 0$. If $x_1 = \pm x_4 \neq 0$, then we get representatives $(1, x, y, 1)$ and $(1, x, y, -1)$.

The group H generated by $e^{t\,\mathrm{ad}X_1}$, $e^{s\,\mathrm{ad}X_4}$ leaves x_1, x_4 fixed and operates on $\mathcal{G}'$ by scalings and hyperbolic rotations which preserve the form $Q(x, y) = x^2 - y^2$ on $\mathcal{G}'$. Thus we get representatives:

$$(1, 0, 0, \epsilon),\ (1, 1, 1, \epsilon),\ (1, 1, -1, \epsilon),\ (1, 1, 0, \epsilon),\ (1, 0, 1, \epsilon) \quad (\epsilon^2 = 1).$$

If $x_1 = \pm x_4 = 0$, then X is in $\mathcal{G}'$ on which the group generated by $e^{t\,\mathrm{ad}X_2}$, $e^{s\,\mathrm{ad}X_3}$ operated trivially. So we get representatives

$$(0, 1, 0, 0),\ (0, 0, 1, 0),\ (0, 1, 1, 0),\ (0, 1, -1, 0).$$

Altogether we have proved that there are 14 conjugacy classes of 1-dimensional subalgebras $\mathfrak{L}_i^1$ of symmetry algebra $\mathcal{G}$ and the representatives of these classes are given by the following subalgebras.

$$\mathfrak{L}_1 = \langle X_1 + \lambda X_4 \rangle \tag{3.1}$$
$$\mathfrak{L}_2 = \langle X_4 + \lambda X_1 \rangle \tag{3.2}$$
$$\mathfrak{L}_3 = \langle X_1 + X_2 + X_3 + X_4 \rangle \tag{3.3}$$
$$\mathfrak{L}_4 = \langle X_1 + X_2 + X_3 - X_4 \rangle \tag{3.4}$$
$$\mathfrak{L}_5 = \langle X_1 + X_2 - X_3 + X_4 \rangle \tag{3.5}$$
$$\mathfrak{L}_6 = \langle X_1 + X_2 - X_3 - X_4 \rangle \tag{3.6}$$
$$\mathfrak{L}_7 = \langle X_1 + X_2 + X_4 \rangle \tag{3.7}$$
$$\mathfrak{L}_8 = \langle X_1 + X_2 - X_4 \rangle \tag{3.8}$$
$$\mathfrak{L}_9 = \langle X_1 + X_3 + X_4 \rangle \tag{3.9}$$
$$\mathfrak{L}_{10} = \langle X_1 + X_3 - X_4 \rangle \tag{3.10}$$
$$\mathfrak{L}_{11} = \langle X_2 + X_3 \rangle \tag{3.11}$$
$$\mathfrak{L}_{12} = \langle X_2 - X_3 \rangle \tag{3.12}$$
$$\mathfrak{L}_{13} = \langle X_2 \rangle \tag{3.13}$$
$$\mathfrak{L}_{14} = \langle X_3 \rangle \tag{3.14}$$

3.2 Optimal system of subalgebras of minimal symmetry algebra of Equation (1.1) for arbitrary $f(u)$

From Section 2, the minimal symmetry algebra $\mathcal{G}$ is 3-dimensional, which is spanned by X_1, X_2, X_3 satisfying the commutation relations

	X_1	X_2	X_3
X_1	0	$-X_3$	$-X_2$
X_2	X_3	0	0
X_3	X_2	0	0

and has the abelian commutator $\mathcal{G}' = \langle X_2, X_3 \rangle$.

Let $X = \sum_{i=1}^{3} x_i X_i$ be an element of $\mathcal{G}$. Writing $X = (x_1, x_2, x_3)$, we see that

$$e^{s\text{ad}X_2} \cdot e^{t\text{ad}X_3}(x_1, x_2, x_3) = (x_1, tx_1 + x_2, sx_1 + x_3).$$

So if $x_1 \neq 0$, then any element $X = (x_1, x_2, x_3)$ is conjugate to $(x_1, 0, 0)$ and as $\mathcal{G}/\mathcal{G}' = \langle X_1 \rangle$ we get one class represented by X_1.

Suppose $x_1 = 0$. Since now $e^{t\text{ad}X_1}$ operates on $\mathcal{G}'$ by hyperbolic rotations (and there are no scalings) we get the following classes in $\mathcal{G}'$.

$$X_2 + \lambda X_3, \quad \lambda X_2 + X_3 \qquad (\lambda \in \mathbb{R})$$

So in all we get the classes

$$\langle X_1 \rangle, \quad \langle X_2 + \lambda X_3 \rangle \quad \langle \lambda X_2 + X_3 \rangle.$$

4 Reductions and invariant solutions corresponding to the optimal system

In this section we give a classification of symmetry reductions of PDE (1.1) by 2-dimensional subalgebras (up to conjugacy) of symmetry algebra. Some exact invariant solutions are also obtained. The reduction procedure explained below is followed.

For each representative X in optimal system of 1-dimensional subalgebras:

(**i**) Reduce the PDE (1.1) to ODE

(**ii**) Determine normalizer of X and conjugacy classes of 1-dimensional subalgebras in the normalizer

The symmetries determined by elements in (**ii**) are inherited by the reduced ODE obtained in (**i**). This allows us to determine all possible reductions and solutions (where possible) by 2-dimensional subalgebras (up to conjugacy).

The reductions in this section require lengthy computations and do not follow a standard algorithmic procedure, so it would be difficult to reproduce them using software. We include a typical computation of reductions in the Case 4.1.1(I) below and omit the details for the remaining cases which can be reproduced in a similar manner.

4.1 Reductions of $u_{tt} = u_{xx} + f(u)$ for $f(u) = u^n$ $n \neq 0, 1$

The equation

$$u_{tt} = u_{xx} + u^n \qquad (n \neq 0, 1) \tag{4.1}$$

admits 4-dimensional symmetry algebra spanned by

$$X_1 = t\frac{\partial}{\partial x} + x\frac{\partial}{\partial t},$$
$$X_2 = \frac{\partial}{\partial t}, \qquad X_3 = \frac{\partial}{\partial x},$$
$$X_4 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{2u}{n-1}\frac{\partial}{\partial u}.$$

Reductions of PDE (4.1) for different cases, associated to 1-dimensional subalgebras are given below.

4.1.1 Subalgebra $\mathfrak{L}_1 = \langle X_1 + \lambda X_4 \rangle$

I. $\lambda \neq 0, \pm 1$

The differential invariants (and hence the similarity variables) for

$$X = X_1 + \lambda X_4 = (\lambda x + t)\frac{\partial}{\partial x} + (x + \lambda t)\frac{\partial}{\partial t} - \frac{2u\lambda}{n-1}\frac{\partial}{\partial u}$$

can be determined by solving the characteristic system

$$\frac{dx}{\lambda x + t} = \frac{dt}{x + \lambda t} = -(n-1)\frac{du}{2u\lambda} \tag{4.2}$$

of $XI = 0$.

Equation (4.2) generates

$$\frac{d(x+t)}{(\lambda+1)(x+t)} = -(n-1)\frac{du}{2u\lambda}$$

and

$$\frac{d(x-t)}{(\lambda-1)(x-t)} = -(n-1)\frac{du}{2u\lambda}$$

which respectively give the similarity variables of $X = X_1 + \lambda X_4$ as

$$\xi(x,t) = \frac{(x+t)^{\frac{1}{\lambda+1}}}{(x-t)^{\frac{1}{\lambda-1}}}$$

$$V(\xi) = u^{n-1} \cdot (x+t)^{\frac{2\lambda}{\lambda+1}}.$$

For reductions we begin by differentiating the similarity variable

$$u^{n-1} \cdot (x+t)^{\frac{2\lambda}{\lambda+1}} = V$$

with respect to t, once and twice, to respectively get

$$(n-1)u^{n-2}u_t(x+t)^{\frac{2\lambda}{\lambda+1}} + u^{n-1} \cdot \frac{2\lambda}{\lambda+1}(x+t)^{\frac{\lambda-1}{\lambda+1}} = V'\xi_t \tag{4.3}$$

and

$$(n-1)(n-2)u^{n-3}u_t^2(x+t)^{\frac{2\lambda}{\lambda+1}} + (n-1)u^{n-2}u_{tt}(x+t)^{\frac{2\lambda}{\lambda+1}}$$
$$+ 2(n-1)u^{n-2}u_t\frac{2\lambda}{\lambda+1}(x+t)^{\frac{\lambda-1}{\lambda+1}} + \frac{2\lambda}{\lambda+1} \cdot \frac{\lambda-1}{\lambda+1}u^{n-1}(x+t)^{\frac{-2}{\lambda+1}} = V''\xi_t^2 + V'\xi_{tt}. \tag{4.4}$$

Similarly differentiating the similarity variable

$$u^{n-1} \cdot (x+t)^{\frac{2\lambda}{\lambda+1}} = V$$

with respect to x, once and twice, respectively gives

$$(n-1)u^{n-2}u_x(x+t)^{\frac{2\lambda}{\lambda+1}} + u^{n-1}\cdot\frac{2\lambda}{\lambda+1}(x+t)^{\frac{\lambda-1}{\lambda+1}} = V'\xi_x \tag{4.5}$$

and

$$\begin{aligned}&(n-1)(n-2)u^{n-3}u_x^2(x+t)^{\frac{2\lambda}{\lambda+1}} + (n-1)u^{n-2}u_{xx}(x+t)^{\frac{2\lambda}{\lambda+1}}\\&+2(n-1)u^{n-2}u_x\frac{2\lambda}{\lambda+1}(x+t)^{\frac{\lambda-1}{\lambda+1}} + \frac{2\lambda}{\lambda+1}\cdot\frac{\lambda-1}{\lambda+1}u^{n-1}(x+t)^{\frac{-2}{\lambda+1}} = V''\xi_x^2+V'\xi_{xx}.\end{aligned} \tag{4.6}$$

Now subtracting Equation (4.6) from Equation (4.4), Equation (4.5) from Equation (4.3) and adding Equation (4.3) & Equation (4.5) leads respectively to the identities

$$\begin{aligned}&(n-1)(n-2)u^{n-3}(x+t)^{\frac{2\lambda}{\lambda+1}}(u_t^2-u_x^2) + (n-1)u^{n-2}(x+t)^{\frac{2\lambda}{\lambda+1}}(u_{tt}-u_{xx})\\&+2(n-1)u^{n-2}\frac{2\lambda}{\lambda+1}(x+t)^{\frac{\lambda-1}{\lambda+1}}(u_t-u_x) = V''(\xi_t^2-\xi_x^2)+V'(\xi_{tt}-\xi_{xx}),\end{aligned} \tag{4.7}$$

$$(x+t)^{\frac{2\lambda}{\lambda+1}}(u_t-u_x) = \frac{1}{(n-1)u^{n-2}}\cdot V'(\xi_t-\xi_x) \tag{4.8}$$

and

$$(x+t)^{\frac{2\lambda}{\lambda+1}}(u_t+u_x) = \frac{1}{(n-1)u^{n-2}}\left(-2u^{n-1}\frac{2\lambda}{\lambda+1}(x+t)^{\frac{\lambda-1}{\lambda+1}} + V'(\xi_t+\xi_x)\right). \tag{4.9}$$

Using Equations (4.8), (4.9) and the relations

$$\begin{aligned}\xi_t &= \frac{2\xi(\lambda x+t)}{(\lambda^2-1)(x^2-t^2)}\\ \xi_x &= \frac{-2\xi(\lambda t+x)}{(\lambda^2-1)(x^2-t^2)}\\ \xi_t+\xi_x &= \frac{2\xi}{(\lambda+1)(x+t)}\\ \xi_t-\xi_x &= \frac{2\xi}{(\lambda-1)(x-t)}\\ \xi_t^2-\xi_x^2 &= \frac{4\xi^2}{(\lambda^2-1)(x^2-t^2)}\\ \xi_{tt}-\xi_{xx} &= \frac{4\xi}{(\lambda^2-1)(x^2-t^2)}\end{aligned}$$

in identity (4.7) yields

$$\begin{aligned}&(n-1)u^{n-2}(x+t)^{\frac{2\lambda}{\lambda+1}}(u_{tt}-u_{xx})(\lambda^2-1)(x^2-t^2)\\&=4\xi^2V''+4\xi V'-8\lambda\xi V'-\frac{n-2}{n-1}4\xi V'\left(\frac{\xi V'}{V}-2\lambda\right).\end{aligned} \tag{4.10}$$

Finally, combining Equations (4.10) and PDE (4.1) leads to

$$\begin{aligned} &4\xi^2 V'' + 4\xi V' - 8\lambda\xi V' - \frac{n-2}{n-1} 4\xi V' \left(\frac{\xi V'}{V} - 2\lambda\right) \\ &= (n-1)u^{n-1}(x+t)^{\frac{2\lambda}{\lambda+1}}(\lambda^2-1)\left(u^{n-1}(x^2-t^2)\right). \end{aligned} \tag{4.11}$$

Now using the relation

$$u^{n-1}(x^2-t^2) = \frac{V}{\xi^{\lambda-1}},$$

we obtain the first reduction of PDE (4.1) to the ODE

$$\xi V'' + V'(1-2\lambda) - \frac{n-2}{n-1} V' \left(\frac{\xi V'}{V} - 2\lambda\right) = \frac{(n-1)(\lambda^2-1)}{4} \frac{V^2}{\xi^\lambda}. \tag{4.12}$$

For second reduction, we use conjugacy classes of the normalizer. The normalizer $N(X_1 + \lambda X_4) = \langle X_1, X_4 \rangle$ is abelian so conjugacy classes are represented by

$$X_1 + cX_4, \quad cX_1 + X_4.$$

The inherited symmetry of ODE (4.12) in both cases is a multiple of

$$\tilde{X} = \xi\frac{\partial}{\partial\xi} + (\lambda-1)V\frac{\partial}{\partial V},$$

so we perform second reduction by $\tilde{X}$. Its similarity variables are

$$r(\xi, V) = \frac{V}{\xi^{\lambda-1}}$$

$$w(r) = \frac{V'}{\xi^{\lambda-2}}$$

which reduce ODE (4.12) to Abel equation of second kind of the form

$$(w + (1-\lambda)r)\frac{dw}{dr} - (1+\lambda)w - \frac{n-2}{n-1}w\left(\frac{w}{r} - 2\lambda\right) = (n-1)(\lambda^2-1)\frac{r^2}{4}.$$

II. $\lambda = 1$.

The similarity variables

$$\xi(x,t) = t - x$$

$$V(\xi) = u^{n-1} \cdot (x+t)$$

of $X = X_1 + X_4$ reduce PDE (4.1) to the ODE

$$4V' + (n-1)^2 V^2 = 0.$$

This can be integrated to get the exact solution

$$u(x,t) = \left\{ \frac{4}{(x+t)[(n-1)^2(t-x)+C]} \right\}^{\frac{1}{n-1}}$$

of PDE (4.1).

III. $\lambda = -1$.

The similarity variables

$$\xi(x,t) = x + t$$
$$V(\xi) = u^{n-1} \cdot (x-t)$$

of $X = X_1 - X_4$ reduce PDE (4.1) to the ODE

$$4V' - (n-1)^2 V^2 = 0.$$

This can be integrated to get the exact solution

$$u(x,t) = \left\{ \frac{4}{(x-t)[-(n-1)^2(x+t)+C]} \right\}^{\frac{1}{n-1}}$$

of PDE (4.1).

IV. $\lambda = 0$

The similarity variables

$$\xi(x,t) = t^2 - x^2$$
$$V(\xi) = u$$

of $X = X_1$ reduce PDE (4.1) to the ODE

$$4\xi V'' + 4V' = V^n. \tag{4.13}$$

For second reduction, we use conjugacy classes of the normalizer of $X = X_1$. The normalizer $N(X_1) = \langle X_1, X_4 \rangle$ is abelian so conjugacy classes are represented by

$$X_1 + cX_4, \qquad cX_1 + X_4.$$

The inherited symmetry of ODE (4.13) in both cases is a multiple of

$$\tilde{X} = \xi \frac{\partial}{\partial \xi} - \frac{V}{n-1} \frac{\partial}{\partial V},$$

so we perform second reduction by $\tilde{X}$. Its similarity variables are

$$r(\xi, V) = \xi^{\frac{1}{n-1}} V$$
$$w(r) = \xi^{\frac{n}{n-1}} V'$$

which reduce ODE (4.13) to Abel equation of second kind of the form

$$4((n-1)w + r)\frac{dw}{dr} = 4w + (n-1)r^n.$$

4.1.2 Subalgebra $\mathfrak{L}_2 = \langle \lambda X_1 + X_4 \rangle$

The cases $\lambda = \pm 1$ reduce to the cases 4.1.1 (II, III).

I. $\lambda = 0$.

The similarity variables

$$\xi(x,t) = \frac{t}{x}$$
$$V(\xi) = u^{n-1} \cdot x^2$$

of $X = X_4$ reduce PDE (4.1) to the ODE

$$(\xi^2 - 1)V'' + 6\xi V' + \frac{n-2}{n-1}\left[\frac{V'^2}{V} - \frac{\xi^2 V'^2}{V} - 4\xi V' - 4V\right] + 6V + (n-1)V^2 = 0. \tag{4.14}$$

For second reduction we use the conjugacy classes

$$X_1 + cX_4, \quad cX_1 + X_4$$

of the normalizer $N(X_4) = \langle X_1, X_4 \rangle$.

The inherited symmetry of ODE (4.14) in both cases is a multiple of

$$\tilde{X} = (1 - \xi^2)\frac{\partial}{\partial \xi} + 2\xi V \frac{\partial}{\partial V},$$

so we perform second reduction by $\tilde{X}$. Its similarity variables

$$r(\xi, V) = V(1 - \xi^2)$$
$$w(r) = (1 - \xi^2)\frac{V'}{V} - 2\xi$$

reduce ODE (4.14) to Bernoulli equation

$$\frac{dw}{dr} + \frac{1}{(n-1)r}w = \left(\frac{4}{(n-1)r} + n - 1\right) w^{-1}$$

which can be solved to get

$$w^2 = 4 + 2\frac{(n-1)^2}{n+1}r + \frac{C}{r^{\frac{2}{n-1}}}.$$

II. $\lambda \neq 0, \pm 1$.

The similarity variables

$$\xi(x,t) = \frac{(x+t)^{\frac{1}{1+\lambda}}}{(x-t)^{\frac{1}{1-\lambda}}}$$

$$V(\xi) = u^{n-1} \cdot (x+t)^{\frac{2}{1+\lambda}}$$

of $X = \lambda X_1 + X_4$ reduce PDE (4.1) to the ODE

$$\xi V'' - V' - \frac{n-2}{n-1} V' \left(\frac{\xi V'}{V} - 2 \right) = \frac{(1-\lambda^2)(n-1)}{4} \frac{V^2}{\xi^{2-\lambda}}. \tag{4.15}$$

For second reduction we use the conjugacy classes

$$X_1 + cX_4, \quad cX_1 + X_4$$

of the abelian normalizer $N(\lambda X_1 + X_4) = \langle X_1, X_4 \rangle$. The inherited symmetry of ODE (4.15) in both cases is a multiple of

$$\tilde{X} = \xi \frac{\partial}{\partial \xi} + (1-\lambda) V \frac{\partial}{\partial V}$$

so we perform second reduction by $\tilde{X}$. Its similarity variables are

$$r(\xi, V) = \frac{V}{\xi^{1-\lambda}}$$

$$w(r) = V' \xi^{\lambda}$$

which reduce ODE (4.15) to following Abel equation of second kind:

$$(w + (\lambda - 1)r) \frac{dw}{dr} - (1+\lambda) w - \frac{n-2}{n-1} w \left(\frac{w}{r} - 2 \right) = \frac{(n-1)(1-\lambda^2)}{4} r^2.$$

4.1.3 Subalgebra $\mathfrak{L}_3 = \langle X_1 + X_2 + X_3 + X_4 \rangle$

The similarity variables

$$\xi(x,t) = t - x$$
$$V(\xi) = (x+t+1) u^{n-1}$$

of $X = X_1 + X_2 + X_3 + X_4$ reduce PDE (4.1) to the ODE

$$4V' + (n-1)^2 V^2 = 0.$$

This yields the solution

$$u(x,t) = \left\{ \frac{4}{(x+t+1)[(n-1)^2(t-x) + C]} \right\}^{\frac{1}{n-1}}$$

of PDE (4.1).

4.1.4 Subalgebra $\mathfrak{L}_4 = \langle X_1 + X_2 + X_3 - X_4 \rangle$

The similarity variables

$$\xi(x,t) = (x-t)e^{x+t}$$
$$V(\xi) = u^{n-1} \cdot (x-t)$$

of $X = X_1 + X_2 + X_3 - X_4$ reduce PDE (4.1) to the ODE

$$4\xi^2 V'' - 4\frac{n-2}{n-1}\xi V' \left(\frac{\xi V'}{V} - 1\right) + (n-1)V^2 = 0. \tag{4.16}$$

The normalizer $N(X) = \langle X_1 - X_4, X_2 + X_3 \rangle$ is abelian so representatives of conjugacy classes are

$$(X_1 - X_4) + c(X_2 + X_3) \quad \text{and} \quad c(X_1 - X_4) + (X_2 + X_3).$$

The inherited symmetry of reduced ODE in both cases is a multiple of

$$\tilde{X} = \xi \frac{\partial}{\partial \xi},$$

so we perform second reduction by $\tilde{X}$. Its similarity variables

$$r(\xi, V) = V$$
$$w(r) = \xi V'$$

reduce the ODE (4.16) to Abel equation of second kind

$$4w\frac{dw}{dr} - 4w - 4\frac{(n-2)}{n-1}w\left(\frac{w}{r} - 1\right) + (n-1)r^2 = 0.$$

4.1.5 Subalgebra $\mathfrak{L}_5 = \langle X_1 + X_2 - X_3 + X_4 \rangle$

The similarity variables of $X = X_1 + X_2 - X_3 + X_4$ are

$$\xi(x,t) = (x+t)e^{x-t}$$
$$V(\xi) = u^{n-1} \cdot (x+t)$$

which reduce PDE (4.1) to

$$4\xi^2 V'' - 4\frac{n-2}{n-1}\xi V' \left(\frac{\xi V'}{V} - 1\right) + (n-1)V^2 = 0. \tag{4.17}$$

The (abelian) normalizer $N(X) = \langle X_1 + X_4, X_2 - X_3 \rangle$ has representatives of conjugacy classes as

$$(X_1 + X_4) + c(X_2 - X_3) \quad \text{and} \quad c(X_1 + X_4) + (X_2 - X_3).$$

In both cases, the inherited symmetry of ODE (4.17) is a multiple of

$$\tilde{X} = \xi \frac{\partial}{\partial \xi}.$$

This, as in Case 4.1.4, reduces ODE (4.17) to Abel equation of second kind

$$4w\frac{dw}{dr} - 4w - 4\frac{(n-2)}{n-1}w\left(\frac{w}{r} - 1\right) + (n-1)r^2 = 0.$$

4.1.6 Subalgebra $\mathfrak{L}_6 = \langle X_1 + X_2 - X_3 - X_4 \rangle$

The similarity variables

$$\xi(x,t) = x + t$$
$$V(\xi) = u^{n-1} \cdot (x - t + 1)$$

of $X = X_1 + X_2 - X_3 - X_4$ reduce PDE (4.1) to

$$4V' - (n-1)^2 V^2 = 0,$$

yielding the solution

$$u(x,t) = \left\{ \frac{-4}{(x-t+1)[(n-1)^2(x+t) + C]} \right\}^{\frac{1}{n-1}}$$

4.1.7 Subalgebra $\mathfrak{L}_7 = \langle X_1 + X_2 + X_4 \rangle$

The similarity variables of $X = X_1 + X_2 + X_4$ are

$$\xi(x,t) = (2x + 2t + 1)e^{2(x-t)}$$
$$V(\xi) = u^{n-1} \cdot (2x + 2t + 1)$$

which reduce PDE (4.1) to

$$16\xi^2 V'' - \frac{n-2}{n-1} 16\xi V' \left(\frac{\xi V'}{V} - 1 \right) + (n-1)V^2 = 0. \tag{4.18}$$

The (abelian) normalizer $N(X) = \langle X_1 + X_2 + X_4, -X_2 + X_3 \rangle$ has representatives of conjugacy classes as

$$(X_1 + X_2 + X_4) + c(-X_2 + X_3) \quad \text{and} \quad c(X_1 + X_2 + X_3) + (-X_2 + X_3).$$

The inherited symmetry of ODE (4.18) in both cases is a multiple of

$$\tilde{X} = \xi \frac{\partial}{\partial \xi},$$

so we perform second reduction by $\tilde{X}$. Its similarity variables

$$r(\xi, V) = V$$
$$w(r) = \xi V'$$

reduce ODE (4.18) to

$$16w\frac{dw}{dr} - 16w - 16\frac{(n-2)}{n-1}w\left(\frac{w}{r} - 1\right) + (n-1)r^2 = 0,$$

which is Abel equation of second kind.

4.1.8 Subalgebra $\mathfrak{L}_8 = \langle X_1 + X_2 - X_4 \rangle$

The similarity variables

$$\xi(x,t) = (2x - 2t + 1)e^{2(x+t)}$$
$$V(\xi) = u^{n-1} \cdot (2x - 2t + 1)$$

of $X = X_1 + X_2 - X_4$ reduce PDE (4.1) to

$$16\xi^2 V'' - 16\frac{n-2}{n-1}\xi V'\left(\frac{\xi V'}{V} - 1\right) + (n-1)V^2 = 0. \tag{4.19}$$

The normalizer $N(X) = \langle X_1 + X_2 - X_4, X_2 + X_3 \rangle$ is abelian and so representatives of conjugacy classes are

$$(X_1 + X_2 - X_4) + c(X_2 + X_3) \quad \text{and} \quad c(X_1 + X_2 - X_4) + (X_2 + X_3).$$

In both cases, the inherited symmetry of ODE (4.19) is a multiple of

$$\tilde{X} = \xi\frac{\partial}{\partial\xi}.$$

So we perform second reduction by $\tilde{X}$ and, like Case 4.1.7, ODE (4.19) reduces to following Abel equation of second kind

$$16w\frac{dw}{dr} - 16w - 16\frac{(n-2)}{n-1}\left(\frac{w}{r} - 1\right) + (n-1)r^2 = 0.$$

4.1.9 Subalgebra $\mathfrak{L}_9 = \langle X_1 + X_3 + X_4 \rangle$

The similarity variables of $X = X_1 + X_3 + X_4$ are

$$\xi(x,t) = (2x + 2t + 1)e^{2(t-x)}$$
$$V(\xi) = u^{n-1} \cdot (2x + 2t + 1)$$

which reduce PDE (4.1) to

$$16\xi^2 V'' - 16\frac{n-2}{n-1}\xi V'\left(\frac{\xi V'}{V} - 1\right) - (n-1)V^2 = 0. \tag{4.20}$$

The normalizer $N(X) = \langle X_1 + X_2 + X_4, -X_2 + X_3\rangle$ has representatives of conjugacy classes as

$$(X_1 + X_2 + X_4) + c(-X_2 + X_3) \quad \text{and} \quad c(X_1 + X_2 + X_4) + (-X_2 + X_3).$$

The inherited symmetry of ODE (4.20) in both cases is a multiple of

$$\tilde{X} = \xi\frac{\partial}{\partial \xi}.$$

Like case 4.1.7, this symmetry reduces ODE (4.20) to

$$16w\frac{dw}{dr} - 16w - 16\frac{(n-2)}{n-1}\left(\frac{w}{r} - 1\right) - (n-1)r^2 = 0.$$

which is Abel equation of second kind.

4.1.10 Subalgebra $\mathfrak{L}_{10} = \langle X_1 + X_3 - X_4\rangle$

The similarity variables

$$\xi(x,t) = (2x - 2t - 1)e^{2(x+t)}$$
$$V(\xi) = u^{n-1}\cdot(2x - 2t - 1)$$

of $X = X_1 + X_3 - X_4$ reduce PDE (4.1) to

$$16\xi^2 V'' - 16\frac{n-2}{n-1}\xi V'\left(\frac{\xi V'}{V} - 1\right) + (n-1)V^2 = 0. \tag{4.21}$$

The normalizer $N(X) = \langle X_1 - X_2 - X_4, X_2 + X_3\rangle$ has representatives of conjugacy classes as

$$(X_1 - X_2 - X_4) + c(X_2 + X_3) \quad \text{and} \quad c(X_1 - X_2 - X_4) + (X_2 + X_3).$$

The inherited symmetry in both cases is a multiple of

$$\tilde{X} = \xi\frac{\partial}{\partial \xi},$$

which, like case 4.1.7, reduces ODE (4.21) to following Abel equation of second kind:

$$16w\frac{dw}{dr} - 16w - 16\frac{(n-2)}{n-1}\left(\frac{w}{r} - 1\right) + (n-1)r^2 = 0.$$

4.1.11 Subalgebra $\mathfrak{L}_{11} = \langle X_2 + X_3 \rangle$

In this case, the similarity variables

$$\xi(x,t) = t - x$$
$$V(\xi) = u$$

yield the trivial solution $u = 0$.

4.1.12 Subalgebra $\mathfrak{L}_{12} = \langle X_2 - X_3 \rangle$

The similarity variables

$$\xi(x,t) = x + t$$
$$V(\xi) = u$$

lead to trivial solution $u = 0$.

4.1.13 Subalgebra $\mathfrak{L}_{13} = \langle X_2 \rangle$

The similarity variables

$$\xi(x,t) = x$$
$$V(\xi) = u$$

of $X = X_2$ reduce PDE (4.1) to

$$V'' + V^n = 0. \tag{4.22}$$

For further reductions, we look at normalizer of X_2 which is $N(X_2) = \langle X_2, X_3, X_4 \rangle$. Like Section 3.2, its representatives of conjugacy classes of 1-dimensional subalgebras are given by

$$cX_2 + X_3, \quad X_2 + cX_3 \quad \text{and} \quad X_4.$$

I. For the symmetries $cX_2 + X_3$ or $X_2 + cX_3$, the inherited symmetry of ODE (4.22) is

$$\tilde{X} = \frac{\partial}{\partial \xi}.$$

Its similarity variables

$$r(\xi, V) = V$$
$$w(r) = V'$$

reduce ODE (4.22) to

$$w\frac{dw}{dr} + r^n = 0,$$

which implies that the solutions of PDE (4.1) can be determined from

$$\left(\frac{du}{dx}\right)^2 = \frac{C - 2u^{n+1}}{n+1}, \qquad \text{if } n \neq -1$$

or

$$\left(\frac{du}{dx}\right)^2 = \ln\frac{C}{r^2}, \qquad \text{if } n = -1$$

II. For the symmetry X_4, the inherited symmetry of ODE (4.22) is

$$\tilde{X} = \xi\frac{\partial}{\partial\xi} - \frac{2V}{n-1}\frac{\partial}{\partial V}.$$

It similarity variables

$$r(\xi, V) = V\xi^{\frac{2}{n-1}}$$
$$w(r) = V'\xi^{\frac{n+1}{n-1}}$$

reduce ODE (4.22) to

$$\left(w + \frac{2r}{n-1}\right)\frac{dw}{dr} - \frac{(n+1)}{n-1}w + r^n = 0$$

which is Abel equation of second kind.

4.1.14 Subalgebra $\mathfrak{L}_{14} = \langle X_3\rangle$

The similarity variables

$$\xi(x,t) = t$$
$$V(\xi) = u$$

of X_3 reduce PDE (4.1) to

$$V'' + V^n = 0. \tag{4.23}$$

The normalizer of X_3 is $N(X_3) = \langle X_2, X_3, X_4\rangle$ and, as above, its representatives of conjugacy classes of 1-dimensional subalgebras are given by

$$cX_2 + X_3, \quad X_2 + cX_3 \quad \text{and} \quad X_4.$$

I. For $cX_2 + X_3$ or $X_2 + cX_3$, the inherited symmetry of ODE (4.23) is

$$\tilde{X} = \frac{\partial}{\partial\xi}.$$

Like case 4.1.13, it implies that the solutions of PDE (4.1) can be determined from

$$\left(\frac{du}{dx}\right)^2 = \frac{C + 2u^{n+1}}{n+1}, \qquad \text{if } n \neq -1$$

or

$$\left(\frac{du}{dx}\right)^2 = \ln Cu^2, \qquad \text{if } n = -1$$

II. For the symmetry X_4, the ODE (4.23) reduces to the Abel equation

$$\left(w + \frac{2}{n-1}r\right)\frac{dw}{dr} - \frac{(n+1)}{n-1}w + r^n = 0,$$

in a manner similar to Case 4.1.13 (II).

4.2 Reductions of $u_{tt} = u_{xx} + f(u)$ for arbitrary $f(u)$

For arbitrary $f(u)$, the equation

$$u_{tt} = u_{xx} + F(u) \tag{4.24}$$

admits 3-dimensional symmetry algebra spanned by

$$X_1 = t\frac{\partial}{\partial x} + x\frac{\partial}{\partial t},$$
$$X_2 = \frac{\partial}{\partial t}, \qquad X_3 = \frac{\partial}{\partial x}.$$

Like Section 4.1, we obtain following reductions through 1-dimensional optimal system of symmetry algebra and classes in the associated normalizers.

4.2.1 Subalgebra $\langle X_2 + \lambda X_3 \rangle$

Like different cases in Section 4.1, the similarity transformations

$$\xi(x,t) = x - \lambda t$$
$$V(\xi) = u$$

of $X = X_2 + \lambda X_3$, followed by the transformations

$$r(\xi, V) = V$$
$$w(r) = V'$$

of inherited symmetry reduce PDE (4.24) to the separable ODE

$$(1-\lambda^2)w\frac{dw}{dr} + f(r) = 0.$$

4.2.2 Subalgebra $\langle \lambda X_2 + X_3 \rangle$

As above, PDE (4.24) to the separable ODE of the form

$$(1-\lambda^2) w \frac{dw}{dr} + f(r) = 0.$$

4.2.3 Subalgebra $\langle X_1 \rangle$

The similarity variables

$$\xi(x,t) = x^2 - t^2$$
$$V(\xi) = u$$

of $X = X_1$, reduce PDE (4.24) to the ODE

$$4\xi V'' + 4V' + f(V) = 0,$$

which can be further analyzed for different forms of $f(u)$.

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References

[1] W. F. Ames, R. J. Lohner and E. Adams, Group properties of $u_{tt} = (f(u)u_x)_x$, Int. J. Non-Linear Mech. **16**, 439 (1981).

[2] G. Baumann, *Symmetry analysis of differential equations with Mathematica* (Springer-Verlag, New York, 2000).

[3] G. W. Bluman and S. Kumei, *Symmetries and differential equations* (Springer-Verlag, New York, 1989).

[4] N. Euler and W. H. Steeb, *Continuous symmetries, Lie algebras and differential equations* (Bibliographisches Institut, Mannheim, 1992).

[5] P. E. Hydon, *Symmetry methods for differential equations* (Cambridge University Press, Cambridge, 2000).

[6] N. H. Ibragimov, Classification of the invariant solutions to the equations of the two-dimensional transint state flow of gas, Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki **7** (4), 19 (1966). English translation: Journal of Applied Mathematics and Technical Physics, **7** (4), 11 (1966).

[7] N. H. Ibragimov, Optimal systems of subgroups and classification of invariant solutions of equations for planar non-stationary gas flows, Master of Science Thesis in Mathematics, Institute of Hydrodynamics, USSR Acad. Sci., Novosibirsk State University (1965). English translation: Paper 1 in N. H. Ibragimov, *Selected Works*, Vol. II, ALGA Publications, Karlskrona, 2006.

[8] N. H. Ibragimov (Editor), *CRC handbook of Lie group analysis of differential equations, Vol. 1, Symmetries, exact solutions and conservation laws* (CRC Press, Boca Raton, 1994).

[9] N. H. Ibragimov (Editor), *CRC handbook of Lie group analysis of differential equations, Vol. 2, Applications in engineering and physical sciences* (CRC Press, Boca Raton, 1995).

[10] N. H. Ibragimov (Editor), *CRC handbook of Lie group analysis of differential equations, Vol. 3, New trends in theoretical developments and computational methods* (CRC Press, Boca Raton, 1996).

[11] N. H. Ibragimov, *Elementary Lie group analysis and ordinary differential equations* (John Wiley & Sons, Chichester, 1999).

[12] Peter E. Clarkson and E. L. Mansfield, Symmetry reductions and exact solutions of a class of nonlinear heat equations, Phys. D. **70** (3), 250 (1994).

[13] E. L. Mansfield, *Differential Gröbner bases* (Ph.D. Thesis, University of Sydney, Australia, 1992).

[14] P. J. Olver, *Applications of Lie groups to differential equations* (Springer-Verlag, New York, 1986).

[15] L. V. Ovsiannikov, *Group analysis of differential equations* (Academic Press, New York, 1982).

[16] P. Rudra, Symmetry group of the non-linear Klein Gordon equation, J. Phys. A: Math. Gen. **19**, 2499 (1986).