Mostow Fibration

Definition 1 A connected subgroup G of $GL(n,\mathbb{R})$ is reductive if its Lie algebra g has a decomposition

$$\mathsf{g} = \mathsf{k} \oplus \mathsf{p}$$

where

(i) $[\mathsf{k},\mathsf{k}] \subset \mathsf{k}, [\mathsf{k},\mathsf{p}] \subset \mathsf{p},\ [\mathsf{p},\mathsf{p}] \subset \mathsf{k}$

(ii) the Lie group $\widetilde{K}$ of $GL(n,\mathbb{C})$ whose Lie algebra is $\widetilde{\mathsf{k}} = \mathsf{k} \oplus i\mathsf{p}$ is compact.

Example 2 *(1) The group $\mathbb{R}^{>0} = \{(r) : r > 0\}$ is reductive, but the isomorphic group $\begin{pmatrix} 1 & \ln r \\ 0 & 1 \end{pmatrix}_{r>0}$ is not reductive.*

(2) The group $SO(n,\mathbb{R}) = \mathsf{k}$ is reductive with $\mathsf{p} = 0$.

(3) The group $SL(n,\mathbb{R}$ is reductive:
$\mathsf{g} = k \oplus p$, where k is the Lie algebra of skew symmetric, p the space of symmetric matrices: here $k \oplus ip$ is the Lie algebra of skew hermitian matrices of trace 0, so it is the Lie algebra of the compact group $SU(n)$.

(4) The group $GL(n,\mathbb{C})$ is reductive:

We have Lie($GL(n,\mathbb{C})$) $= \mathsf{k} \oplus i\mathsf{k}$, where k is the Lie algebra of unitary matrices. Embed $M(n,\mathbb{C})$ in $M(2n,\mathbb{R})$ by $A + iB \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$ etc.

Proposition 3 *(i) The group G is a closed subgroup of $GL(n,\mathbb{R})$ and $G = KP$, where K is generated by $\exp(X) : X \in \mathsf{k}$ and $P = \exp(\mathsf{p})$.*

(ii) There is a $\widetilde{K}$-invariant hermitian inner product on $\mathbb{C}^n$ which is real-valued on $\mathbb{R}^n$ and on orthonormal basis of $\mathbb{R}^n$ remains an orthonormal basis of $\mathbb{C}^n$. (see Indag. Math. N.S. 10(4), 473–483).

The group $\widetilde{K}$ is represented by unitary matrices, therefore k is represented by real skew-symmetric matrices and p by real symmetric matrices.

The form $B(X,Y) = \mathrm{Tr}(XY)$ *is non-degenerate; it is negative definite on* k *and positive definite on* p.

The main technical tool in Mostow [ ] is a generalization of the polar decomposition. For this, he uses the geometry of the symmetric space $GL(n,R)/O(n,R)$. Put $G = GL(n,R),\ K = O(n,R)$. By polar decomposition, $G = KP$. To $G/K = P$, we give the G-invariant metric as follows: Put $\xi_0 = eK$. We can identify the tangent space at ξ_0 with the vector space of all symmetric matrices: p. If $v \in \mathsf{p}$, then $e^{tv} \cdot \xi$ is a curve with $d/dt|_{t=0}(e^{tv} \cdot \xi_0) = v$.

The map from $G \to P$, $g \mapsto g^t$ factorizes through K. The G-invariant action on P is therefore $g \cdot x = gxg^t$. The metric on $T_{\xi_0}(G/K) = T_e(P)$ is $\|\vec{v}\|^2 = \mathrm{Tr}(\vec{v} \cdot \vec{v})$, which is K-invariant.

Now if $p \in P$ and $\vec{w} \in T_p(P)$, then as $p = qq^t = q^2$ for some q,

$$\begin{aligned} \|\vec{w}\|^2 &= \mathrm{Tr}(q^{-1}\vec{w}(q^{-1})^t)^2 \\ &= \mathrm{Tr}(q^{-1}\vec{w}q^{-1} \cdot q^{-1}\vec{w}q^{-1}) \\ &= \mathrm{Tr}(q^{-1}\vec{w}q^{-2}\vec{w}q^{-1}) \\ &= \mathrm{Tr}(q^{-2}\vec{w}q^{-2}\vec{w}) = \mathrm{Tr}(p^{-1}\vec{w})^2 = \|p^{-1}\vec{w}\|^2. \end{aligned}$$

Therefore, if $\gamma(t)$ is a curve in P, then

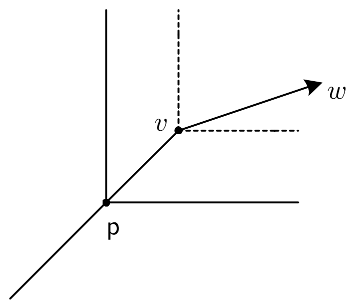
$$\left(\frac{ds}{dt}\right)^2 = \mathrm{Tr}[\gamma(t)^{-1}\gamma'(t)]^2.$$

Now $G/K = P$ is a symmetric space of curvature ≤ 0.

Such spaces have the following property.

Theorem 4 *If M is a complete Riemannian manifold of non-positive curvature, then for all $p \in M$, $v \in T_p(M)$ and $w \in T_v(T_p(M))$, one has the inequality*

$$\|d\exp_p(v)(w)\| \geq \|w\|$$



(see Indag. Math. paper cited earlier)
(Mostow gives a proof from first principles).

In particular, for any curve $\{\gamma(t)\} \subseteq T_p(M)$, we have

$$\mathrm{length}(\exp_p \circ(\gamma)) \geq \mathrm{length}(\gamma).$$

Let p = the space of all symmetric matrices. $P = \exp(\mathsf{p})$ is the space of all positive definite matrices, with the Riemannian metric defined above. Since P is homeomorphic to p, it is a complete space of curvature ≤ 0.

Proposition 5 *For $p \in P$, $\exp(t \log p)$, $0 \leq t \leq 1$ is the unique geodesic in P joining the identity e to p.*

Proof. Let $H = \log p$. Now, if $f(t) = e^{tH}$, then $f'(t) = He^{tH}$, so $\|\dot{f}(t)\|^2 = \mathrm{Tr}(e^{-tH}He^{tH})^2 = \mathrm{Tr}(H^2)$. So $|\dot{f}(t)| = \|H\|$. Therefore

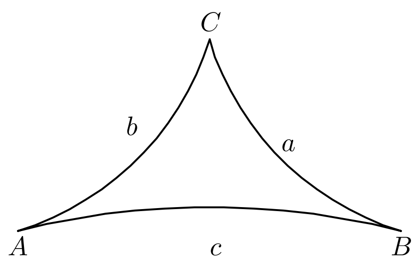
$$\int_0^1 \|\dot{f}(t)\|dt = \|t\| = \mathrm{dist}(H, 0) = \mathrm{dist}(\log p, \log e) \quad (e = \text{ identity of } G).$$

Since $\|H\| \leq$ length of any path joining H to $0 \leq$ length of any path in P joining $\exp(H)$ with $\exp(0)$, we see that the path $f(t) = e^{tH}$, $0 \leq t \leq 1$ is the unique geodesic joining e with p (because it is a constant speed curve).

By homogeneity, this is true for any two points (this also follows at once from Cartan-Hadamard). ∎

Proposition 6 *The Riemannian angle between any two paths f and g intersecting at e ($e =$ identity) is equal to the euclidean angle between the paths* $\log f$ *and* $\log g$ *intersecting at* 0.

Moreover, in any geodesic triangle



we have

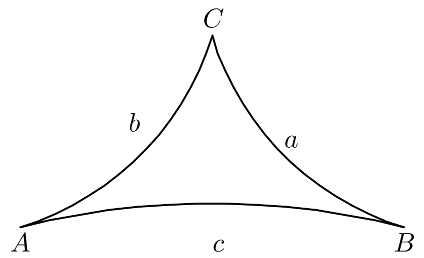
$$c^2 \geq a^2 + b^2 - 2ab\cos\widehat{C}.$$

Proof. By Proposition 1, the usual exponential map from p to P is the Riemannian exponential map of $T_e(P) = p$ onto P. If $f(t) = \exp(\varphi(t))$, then $f'(t) = d\exp_{\varphi(t)}(\varphi'(t))$, so if $f(0) = e$, then $\varphi(0) = 0$ and $f'(0) = d\exp_0(\varphi'(0)) = \varphi'(0)$. Therefore, the angle between the curves $f(t) = e^{\varphi(t)}$, $g(t) = e^{\psi(t)}$ at $t = 0$ is the same as the angle between $e^{tf'(0)}$ and $e^{tg'(0)}$.

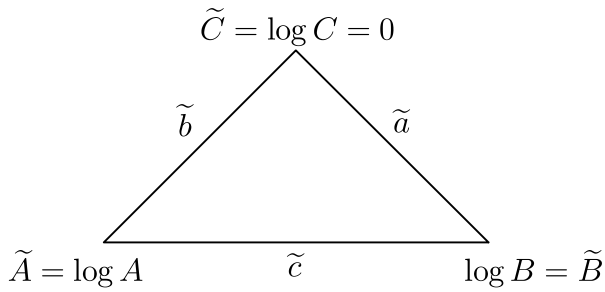
Now, $\langle f'(0), g'(0)\rangle = \mathrm{Tr}(f'(0)\cdot g'(0)) = \mathrm{Tr}(\varphi'(0)\cdot\psi'(0))$. So the angle of intersection between f and g at $e =$

$$\text{angle between } \log f(t) \text{and} \log g(t) \quad \text{at } t = 0.$$

Take a geodesic triangle



Since the G-action $g \cdot x = gxg^t \quad (x \in P)$ is transitive, we may suppose that $C = e$ (identity of G). We compare this with the triangle



By Proposition 1, $\widetilde{a} = a$, $\widetilde{b} = b$ and by what was shown in Proposition 6. $\hat{C} = \hat{\hat{C}}$.

By the distance increasing property of the exponential map on spaces of curvature ≤ 0, we see that $C^2 \geq (\widetilde{C})^2$. Therefore,

$$\begin{aligned} C^2 \geq (\widetilde{C})^2 &= (\widetilde{a})^2 + (\widetilde{b})^2 - 2\widetilde{a}\widetilde{b} \cos \widehat{\widetilde{C}} \\ &= a^2 + b^2 - 2ab \cos \widehat{C} \end{aligned}$$

So

$$\boxed{C^2 \geq a^2 + b^2 - 2ab \cos \widehat{C}}.$$

■

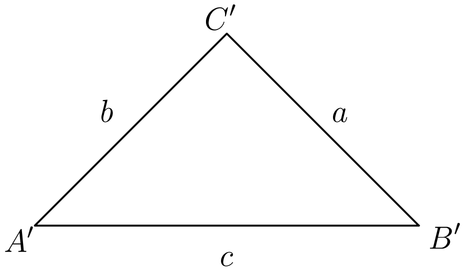
Proposition 7 *The sum of angles in a geodesic triangle is* $\leq 2\pi$.

Proof. By the cosine law

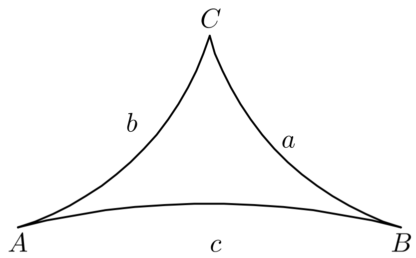
$$c^2 \geq a^2 + b^2 - 2ab \cos \widehat{C} \geq a^2 + b^2 - 2ab = (a-b)^2.$$

If $a \geq b$, then $c \geq a - b$, so $c + b \geq a$. If $a \leq b$, then $c + b \geq a$. In any case $a \leq b + c$.

Construct an euclidean triangle with sides a, b, c:



Compare it with



So

$$c^2 = a^2 + b^2 - 2ab\cos\widehat{C}' \geq a^2 + b^2 - 2ab\cos\widehat{C}.$$

Hence $\cos\widehat{C}' \leq \cos\widehat{C}$, so $\widehat{C}' \geq \widehat{C}$.

Similarly, $\widehat{A}' \geq \widehat{A}$, $\widehat{B}' \geq \widehat{B}$. Hence $\widehat{A}' + \widehat{B}' + \widehat{C}' \geq \widehat{A} + \widehat{B} + \widehat{C}$, i.e., $2\pi \geq \widehat{A} + \widehat{B} + \widehat{C}$.

■

For notational convenience, from now on $\widetilde{G} = Gl(n, R)$, $\widetilde{K} = O(n, R)$. So $\widetilde{G} = \widetilde{K}\widetilde{P}$. G is a reductive subgroup of $\widetilde{G}$.

By the proposition on p. 1, we have a compatible decomposition $G = KP$ where K is a closed subgroup of $\widetilde{K}$ and $P = \exp(\mathsf{p}) \subset \widetilde{P}$ and $[\mathsf{p}, \mathsf{p}] \subset \mathsf{k}$, $[\mathsf{k}, \mathsf{p}] \subset \mathsf{p}$.

Proposition 8 $\exp(\mathsf{p})$ *is a totally geodesic subspace of* $\exp(\widetilde{\mathsf{p}})$*, where* $\widetilde{\mathsf{p}}$ *is the space of all symmetric matrices.*

Proof. The geodesic joining e to $\exp(X)$ $(X \in \mathsf{p})$ is $\{\exp(tx)\}_{0\leq t\leq 1}$. For a fixed $a \in p = \exp(\mathsf{p})$, the map $f \mapsto afa$ maps P to P and it has an inverse $f \mapsto a^{-1}fa^{-1}$ so the map $f \mapsto afa$ is $1:1$ and onto P.

Recalling that G operates on P by $g{\cdot}x = gxg^t$ and this action preserves the metric on P, we see that the geodesic $\{\exp(tX)\}_{0\leq t\leq 1}$ is mapped to the geodesic $\{a\exp tXa\}_{0\leq t\leq 1}$ which joins a^2 to $ae^{tX}a$. Since every element of $\exp(\mathsf{p})$ can be written as a^2 for some $a \in \exp(\mathsf{p})$, and $f \mapsto afa$ is surjective, we see that $\exp(\mathsf{p})$ is a totally geodesic subspace of $\exp(\widetilde{p})$. ■

Proposition 9 *Let $F = \mathsf{p}^{\perp}$. Then*

$$\exp(\widetilde{\mathsf{p}}) = \{efe : e \in \exp(\mathsf{p}), f \in \exp(\mathsf{p}^{\perp})\}.$$

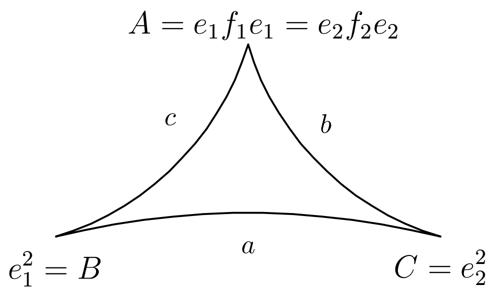
Proof. Step 1: Define

$$\varphi : E \times F \to \exp(\widetilde{\mathsf{p}})$$

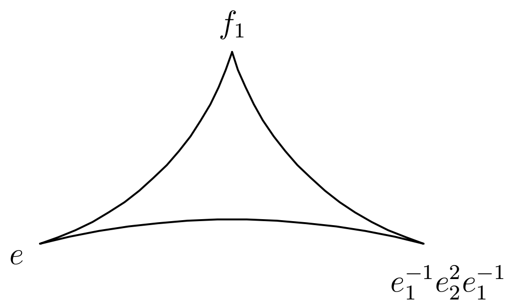
$(E = \mathsf{p})$ by

$$\varphi(e, f) = efe.$$

Suppose $e_1 f_1 e_1 = e_2 f_2 e_2$. Consider the triangle



By the isometry $x \mapsto e_1^{-1} x e_1^{-1}$, this is mapped onto



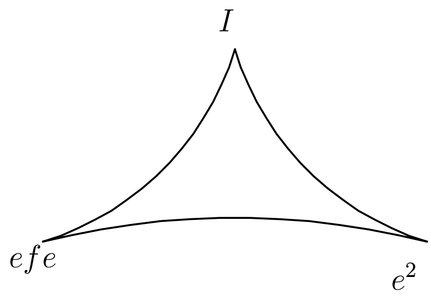
Denote by $\overline{[x, y]}$ the geodesic segment joining x and y. So, by Proposition 8, $\overline{[e, e_1^{-1} e_2^2 e_1^{-1}]}$ is contained in $\exp(\mathsf{p})$ and $\overline{[e, f]}$ is contained in $\exp(\mathsf{p}^{\perp})$. Therefore, by Proposition 6, the angle at vertex $e = 90°$, so the angle at vertex $B = 90°$. Similarly, the angle at vertex $C = 90°$. Hence, by the cosine law,

$$b^2 \geq a^2 + c^2, \quad c^2 \geq b^2 + a^2.$$

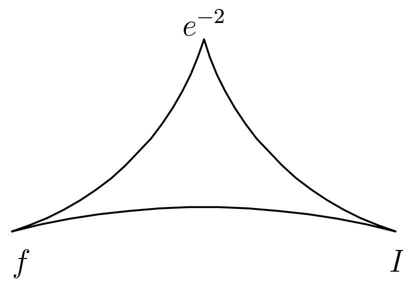
So $b^2 = c^2$ and $a^2 = 0$. Hence $e_1^2 = e_2^2$, so $e_1 = e_2$. Therefore $f_1 = f_2$. This means that φ is $1:1$. ■

Step 2: Im φ is closed. We estimate dist.$(efe, I) = d(efe, I)$ in terms of $d(e, I)$ and $d(f, I)$.

Consider the geodesic triangle



which is isometric to



Now $\widehat{I} = 90°$, so $\hat{e}^2 = 90°$. So by the cosine law

$$\begin{aligned} [d(efe, I)]^2 &\geq [d(efe, e^2)] + [d(e^2, I)]^2 \\ &= [d(f, I)]^2 + [2d(e, I)]^2. \end{aligned}$$

So $d(efe, I) \geq \max\{d(f, I), d(e, I)\}$.

Suppose $e_n f_n e_n \to x \in \exp(\widetilde{\mathsf{p}})$. So $d(e_n, f_n e_n, I) \to d(x, I)$. So as $d(e_n, I), d(f_n, I) \leq d(e_n f_n e_n, I)$, we see that $\{e_n\}$, $\{f_n\}$ are bounded.

By extracting convergent subspaces, we see that $e_n f_n e_n$ converges to $efe = x$. Hence Im φ is closed.

Step 3: φ is an open map. Since φ is continuous and $1:1$ and $E \times F$ and P are euclidean spaces of the same dimension, φ maps open sets to open sets. As im φ is closed, we must have image $\varphi = P$. Hence $\varphi : E \times F \to P$ is a homeomorphism.

Proposition 10 *Any non-singular $n \times n$-matrix can be expressed uniquely and continuously as $k \cdot f \cdot e$ where k is orthogonal and $e \in \exp(\mathsf{p}), f \in \exp(\mathsf{p}^\perp)$.*

Proof. Given a non-singular matrix x, $x^t x$ is positive and symmetric so it belongs to $\exp(\widetilde{p})$. Hence we can find $f \in \exp(p^\perp)$ so that

$$x^t x = ef^2 e.$$

Note that if $x = kfe$, then $x^t = efk^{-1}$, so $x^t x = ef^2 e$. So we set $k = xe^{-1}f^{-1}$. Then $k^t = f^{-1}e^{-1}x^t$ and

$$k^t k = f^{-1}e^{-1}x^t x e^{-1} f^{-1} = f^{-1}e^{-1}(ef^2 e)e^{-1}f^{-1} = I.$$

Now if $x = k_1 f_1 e_1 = k_2 f_2 e_2$, then $x^t x = e_1 f_1^2 e_1 = e_2 f_2^2 e_2$, so $e_1 = e_2, f_1 = f_2$ and $k_1 = k_2$. Hence the map $\theta : (k, f, e) \mapsto kfe$ is $1:1$ and onto.

In the representation $x = kfe$, e and f depend continously on $x^t x$, so on x and therefore k also depends continously on x. Therefore θ^{-1} is also continuous. ■

The Mostow Fibration: We have

$$\widetilde{G} = \widetilde{K} F E$$

and $G = KE$, where G is a reductive subgroup of $\widetilde{G}$. We define a map

$$\widetilde{K} \underset{K}{\times} F \to \widetilde{G}/G$$

by $\widetilde{k} \times f \mapsto \widetilde{k} f G$, which is surjective as $\widetilde{G} = \widetilde{K} F E$. Now if $\widetilde{k} f G = \widetilde{k}_1 f_1 G$, then $\widetilde{k} f = \widetilde{k}_1 f_1 k e = \widetilde{k}_1 k (k^{-1} f_1 k) e$.

Since K maps p onto p (i.e. $kzk^{-1} \in \mathsf{p}$ if $z \in \mathsf{p}$), we see that it also maps $p^\perp$ to $p^\perp$. Therefore, by the uniqueness of the decomposition given in Proposition 10, we see that

$$\widetilde{k} = \widetilde{k}_1 k, \;\; f = k^{-1} f_1 k, \;\; e = I.$$

So,

$$\widetilde{k}_1 = \widetilde{k} k^{-1}, \;\; f_1 = k f k^{-1}.$$

Hence the map

$$\begin{aligned} \widetilde{K} \underset{K}{\times} F &\to \widetilde{G}/G \\ [\widetilde{k} \times f] &\mapsto \widetilde{k} f G \end{aligned}$$

is a diffeomorphism.

Remark 11 The same proof works if $\widetilde{G} \supset G$ is a reductive pair (for any $\widetilde{G}$, G with compatible decompositions).

In particular, this applies to $K^{\mathbb{C}}/L^{\mathbb{C}}$:

$$k^{\mathbb{C}} = k \oplus ik, \quad \ell^{\mathbb{C}} = \ell \oplus i\ell$$

So

$$\begin{array}{c} K \underset{L}{\times} \exp(i\ell^\perp) \cong K^{\mathbb{C}}/L^{\mathbb{C}} \\ \downarrow \\ K/L \end{array}$$

In this sense, the affine quadratic is real-analytically a vector bundle over the real sphere.