

1 P3 Integer and Rational Number Exponent

Definition of Natural Number Exponents

If b is any real number and n is any natural number, then $b^n = b \cdot b \cdot b \cdots b$. In the expression b^n , b is the **base**, n is the **exponent**, and b^n is the **nth power** of b .

Example 1 Find the value of 1) 3^4 2) $(-5)^4$ 3) -5^4

Definition of b^0

For any nonzero real number b , $b^0 = 1$.

Example 2 Find the value of 1) 7^0 2) $(-3)^0$ 4) $(a^2 + 1)^0$

Definition of b^{-n}

If $b \neq 0$ and n is any natural number, then $b^{-n} = \frac{1}{b^n}$ and $\frac{1}{b^{-n}} = b^n$

Example 3 Find the value of 1) 3^{-2} 2) $\frac{1}{4^{-3}}$ 3) $\frac{5^{-2}}{7^{-1}}$

Restriction Agreement

The expression 0^0 , 0^n where n is a negative integer, and $\frac{x}{0}$ are all undefined expression. Therefore, all values of variables in this text are restricted to avoid any one of these expressions.

Example 4 What are the restrictions in the following expression $\frac{x^0(y+2)^{-3}}{z-4}$

Properties of Exponents

If m , n , and p are integers and a and b are real numbers, then

Product $b^m \cdot b^n = b^{m+n}$ Quotient $\frac{b^m}{b^n} = b^{m-n}$ Power $(b^m)^n = b^{mn}$
 $(a^m b^n)^p = a^{mp} b^{np}$ $\left(\frac{a}{b}\right)^p = \frac{a^p}{b^p}$, $b \neq 0$.

To simplify an expression involving exponents, write the expression in a form in which each base appears at most once and no powers of powers or negative powers.

Example 5 Simplify 1) $\left(\frac{2abc^2}{5a^2b}\right)^3$ 2) $\frac{x^n y^{2n}}{x^{n-1} y^n}$ 3) $(2x^{-3}y^0)(3^{-1}xy)^2$
4) $\left(\frac{x^{4-n}y^{n+4}}{xy^{n-4}}\right)^2$ 5) $\frac{(2x^{-2}y^3)^{-2}(-4x^4y^{-1})^3}{-8x^{-2}y^2}$

Scientific Notation

A number written in scientific notation has the form $a \cdot 10^n$, where n is an integer and $1 \leq a < 10$.

Procedure to change a number from its decimal form to scientific notation

1) For numbers greater than 10, move the decimal point to the position to the right of the first digit. The exponent n will equal the number of places the decimal point has been moved.

2) For numbers less than 1, move the decimal point to the position to the right of the first nonzero digit. The exponent n will be negative, and its absolute value will equal the number of places the decimal point has been moved.

Example 6 Write each number in scientific notation

1) 7,430,000 2) 0.00000078 3) 980.665×10^{-2}

Procedure to change a number from scientific notation to its decimal form

1) If the exponent is positive, move the decimal point to the right the same number of places as the exponent.

2) If the exponent is negative, move the decimal point to the left the same number of places as the absolute value of the exponent.

Example 7 Change each number from scientific notation to decimal notation

1) 3.5×10^5 2) 2.51×10^{-8}

Rational Exponents and Radicals

Definition of $b^{\frac{1}{n}}$

If n is an even positive integer and $b \geq 0$, then $b^{\frac{1}{n}}$ is the nonnegative real number such that $\left(b^{\frac{1}{n}}\right)^n = b$.

If n is an odd positive integer, then $b^{\frac{1}{n}}$ is the real number such that $\left(b^{\frac{1}{n}}\right)^n = b$.

Example 8 Find the value of the following (if possible) 1) $25^{\frac{1}{2}}$ 2) $(-64)^{\frac{1}{3}}$ 3) $-49^{\frac{1}{2}}$ 4) $(-16)^{\frac{1}{2}}$

Definition of $b^{\frac{m}{n}}$

For all positive integers m and n such that $\frac{m}{n}$ is in simplest form, and for all real numbers b for which $b^{\frac{1}{n}}$ is a real number $b^{\frac{m}{n}} = \left(b^{\frac{1}{n}}\right)^m = \left(b^m\right)^{\frac{1}{n}}$.

Example 9 Find the value of $8^{\frac{4}{3}}$ in two ways.

The same properties of integer exponents applied to rational exponents given that everything is defined.

Example 10 Simplify 1) $\left(\frac{a^3 b^4}{8a^6 b}\right)^{\frac{2}{3}}$ 2) $\left[\frac{(5x^2 y)^{-1} (5x^3 y^{-2})^2}{5(xy)^{-3} (x^5 y^{-2})^{-1}}\right]^{\frac{-1}{4}}$ 3) $\left[\frac{\left(a^{\frac{1}{6}} b^{\frac{2}{3}}\right) \left(b^{\frac{-1}{2}} a^{\frac{1}{2}}\right)}{a^{-2} b^{\frac{1}{2}}}\right]^3$

Example 11 If $m = 8$ and $n = 4$, then find the value of $\left[\frac{3m^{\frac{8k}{3}}}{n^{\frac{3}{4}}}\right]^2 \left[\frac{8n^3}{m^6}\right]^{\frac{2}{3}}$

Simplify Radical Expression

We express Radicals by the notation $\sqrt[n]{b}$. The number b is the **radicand** and the positive integer n is the **index** of the radical. $\sqrt[n]{b}$ is called the **n th root** of b .

Definition of $\sqrt[n]{b}$

If n is a positive integer and b is a real number such that $b^{\frac{1}{n}}$ is a real number, then $\sqrt[n]{b} = b^{\frac{1}{n}}$.

If $n = 2$, then $\sqrt{b} = \sqrt{b}$ and it is called the **principle square root** of b .

Example 12 Find the value of 1) $\sqrt{25}$ 2) $-\sqrt{25}$ 3) $\sqrt{-25}$ 4) $\sqrt[3]{-27}$

Definition of $(\sqrt[n]{b})^m$

For all positive integers n , all integers m , and all real numbers b such that $\sqrt[n]{b}$ is a real number, then $(\sqrt[n]{b})^m = \sqrt[n]{b^m} = b^{\frac{m}{n}}$.

Note that we use the denominator n as the index of the radical and the numerator m as the power of the radicand or the power of the radical.

Example 13 Write the radical expression $\sqrt{(2ab)^3}$ in exponential form

Example 14 Write the exponential expression $(5xy)^{\frac{2}{3}}$ in radical form.

Example 15 Find $\sqrt{x^2}$ when $x = 5$ and $x = -5$.

Example 16 Find $\sqrt[3]{x^3}$ when $x = 5$ and $x = -5$.

Definition of $\sqrt[n]{b^n}$

If n is an even natural number and b is a real number, then $\sqrt[n]{b^n} = |b|$.

If n is an odd natural number and b is a real number, then $\sqrt[n]{b^n} = b$.

Example 17 Simplify 1) $\sqrt[4]{16z^4}$ 2) $\sqrt[5]{32a^5}$ 3) $\sqrt{y^4}$

Example 18 If $x < y$, then simplify the expression $[(x^2 - 2xy + y^2)]^{\frac{1}{2}} + [(x - y)^3]^{\frac{1}{3}}$.

Properties of Radicals

If m and n are natural numbers and a and b are nonnegative real numbers, then

Product $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$ Quotient $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}, b \neq 0$ Index
 $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$

Example 19 If $\sqrt[4]{x} = m$ and $\sqrt[3]{x} = n$, then write $\sqrt[5]{mn}$ in terms of x .

Example 20 If $x \geq 0$, then simplify $\sqrt{x^3 \sqrt{x^5}}$

Example 21 If $m > 0$, then simplify $\frac{\sqrt[3]{m^5} \sqrt[6]{m^3}}{\sqrt[4]{\sqrt[3]{m^2}}}$

A radical is in **simplest form** if it satisfies the following conditions:

- 1) The radicand contains only powers less than the index. $(\sqrt{x^5})$
- 2) The index of the radical is as small as possible. $(\sqrt[9]{x^3})$
- 3) The denominator has been rationalized where no radicals appear in the denominator.
- 4) No fractions appear under the radical sign.

Example 22 Simplify 1) $\sqrt{18x^4y^3}$ 2) $\sqrt[3]{-16x^4y^6}$

Like radicals have the same radicand and the same index.

Addition We add only like radicals

Example 23 Simplify 1) $3\sqrt[3]{5xy^2} - 4\sqrt[3]{5xy^2}$ 2) $3\sqrt[3]{x} + 5\sqrt[3]{x}$ 3) $2\sqrt{3} + 6\sqrt{5}$ 4) $5x\sqrt[3]{16x^4} - \sqrt[3]{128x^7}$

Multiplication

Example 24 Perform the indicated operation 1) $(\sqrt{3} + 5)(\sqrt{6} - 2)$ 2) $(2\sqrt{3} - 3)^2$

Rationalizing

Example 25 Rationalize the denominator of the following:

1) $\frac{1}{\sqrt{x}}$ 2) $\frac{5}{\sqrt[3]{a}}$ 3) $\sqrt{\frac{3}{32y}}$ 4) $\frac{1}{\sqrt{x^3}}$ 5) $\frac{1}{\sqrt{x} - \sqrt{y}}$ 6) $\frac{\sqrt{3} + \sqrt{2}}{2\sqrt{3} - 3\sqrt{2}}$ 7) $\frac{5}{(2 - \sqrt{3})(1 + \sqrt{2})}$

Example 26 If $a^{\frac{2}{5}}a^p = a^2$, then find p .

Example 27 If $2^x = y$, then find 2^{x-3} in terms of y .

Example 28 Rationalize the numerator of $\frac{\sqrt{a+h} - \sqrt{a}}{h}$.

Example 29 Simplify $\frac{2}{\sqrt[3]{54}} + \frac{4}{\sqrt[3]{16}} - \frac{1}{\sqrt[3]{2}}$.